

CLASSIFICATION THEORY FOR THEORIES WITH
NIP - A MODEST BEGINNING
SH715

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ABSTRACT. A relevant thesis is that for the family of complete first order theories with NIP (i.e. without the independence property) there is a substantial theory, like the family of stable (and the family of simple) first order theories. We examine some properties.

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ANOTATED CONTENT

§1 Indiscernible sequences and averages

[We consider indiscernible sequence $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ wondering, do they have an average as in the stable case. We investigate the set of $\varphi(\bar{x}, \bar{y})$ such that every instance $\varphi(\bar{x}, \bar{a})$ divide $\bar{\mathbf{b}}$ to a finite/co-finite sets, and some can divide it only to finitely many intervals; this is always the case if T has NIP. If T has NIP, indiscernible sequences behave reasonably while indiscernible sets behave nicely and so does $p \in S(M)$ connected with them which we call stable types. We then investigate having an unstable/nip $\varphi(x, y; \bar{c})$, i.e. on singletons.]

§2 Characteristics of types

[Each indiscernible sequence $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$, has for each $\varphi = \varphi(\bar{x}, \bar{y})$ a characteristic number $n = n_{\bar{\mathbf{b}}, \varphi}$, the maximal number of intervals to which an instance $\varphi(\bar{x}, \bar{b})$ can divide $\bar{\mathbf{b}}$. We wonder what we can say about it.]

§3 Shrinking indiscernibles

[For an indiscernible sequence to a set, if we increase the set a little, not much indiscernability is lost.]

§4 Perpendicular endless indiscernible sequences

[We define perpendicularity and investigate its basic property; any two mutually indiscernible sequences are perpendicular. E.g. (for NIP theories) one sequence cannot be non perpendicular to $\geq |T|^+$ pairwise perpendicular sequences. We then deal with $\mathbf{F}_{|T|^+}^{\text{sp}}$ -constructions.]

§5 Indiscernible sequences perpendicular to cuts

[Using construction as above we show that we can build models controlling quite tightly the dual cofinality of such sequences.]

§1 INDISCERNIBLE SEQUENCES AND AVERAGES

We try to continue [Sh:c, Ch.II,4.13], but we do not rely on it. NIP stands for = no independence property. For stable (complete first order) theories, the notions of indiscernible set and its average (and local versions of them) play important role. For unstable theory, indiscernible sequences are not necessarily indiscernible sets. Still for indiscernible sets \mathbf{I} if T has NIP, the basic claim guaranteeing the existence of averages (any $\varphi(\bar{x}, \bar{b})$ divide \mathbf{I} to a finite and co-finite set) hold, and for indiscernible sequences the division is into the union of $<_{n_\varphi}$ convex sets. For any T , we can still look at the first order formulas $\varphi(\bar{x}, \bar{y})$ which behaves well, i.e. any $\varphi(\bar{x}, \bar{b})$ divide any appropriate \mathbf{I} as above.

In 1.3 - 1.7 + 1.7(c) + (d) we define the relevant notions: average ($\text{Av}(\mathbf{J}, D)$ or $\text{Av}_\varphi(\mathbf{J}, D)$ or $\text{Av}(\mathbf{J}, (\bar{b}_t : t \in I))$) and av for finite sequence averaging formulas for $\mathbf{I}(\text{avf}, \text{daf})$, and state some basic properties.

In particular we look at indiscernible sequence of “finite distance”, those are related to canonical bases (of types, of indiscernible sets) play important role for stable theories, hence we try to define parallels in 1.7, see 1.10(2).

Next we note a dichotomy for the types $p \in S^m(M)$. Such a type p may be stable (see Definition 1.15, Claim 1.11 - 1.13); not only is the type definable, but for every ultrafilter D on ${}^m M$ with $\text{Av}(M, D) = p$, any indiscernible set constructed from D is an indiscernible set, and the definition comes from appropriate finite large enough (Δ, k) -indiscernible sets. If $p \in S^m(M)$ is not stable, then there is a partial order with infinite chains closely related to it. We conclude that if T is unstable (with NIP), then some $\varphi(x, y, \bar{c})$ define a quasi order with infinite chains and if T is unstable some $\varphi(x, y; \bar{c})$ has the order property (though not necessarily the property (E) of Ehrenfeucht).

1.1 Context. T a complete first order theory, its monster model being $\mathfrak{C} = \mathfrak{C}_T$ as usual in [Sh:c].

1.2 Definition. T is NIP means it does not have the independence property, IP in short, i.e. for no $\varphi(\bar{x}, \bar{y})$ do we have for every n

$$\boxtimes_\varphi^n \mathfrak{C} \models (\exists \bar{y}_0, \dots, \bar{y}_{n-1}) \bigwedge_{\eta \in {}^n 2} (\exists \bar{x}) \left(\bigwedge_{\ell < n} \varphi(\bar{x}, \bar{y}_\ell)^{\text{if}(\eta(\ell))} \right).$$

1.3 Definition. 1) For $\mathbf{J} \subseteq {}^\omega \mathfrak{C}$, $m < \omega$, a set $\mathbf{I} \subseteq {}^m \mathfrak{C}$, an ultrafilter on D over \mathbf{I} we let $\text{Av}(\mathbf{J}, D)$ be $\{\varphi(\bar{x}, \bar{a}) : \bar{x} = \langle x_\ell : \ell < m \rangle, \bar{a} \in \mathbf{J} \text{ and } \{\bar{b} \in \mathbf{I} : \models \varphi(\bar{b}, \bar{a})\} \in D\}$. It will be called the D -average over \mathbf{J} . If $\mathbf{J} = {}^\omega B$ we may write B instead of \mathbf{J} (or M if $B = |M|$). (Av stands for average).

2) If D is an ultrafilter over $\mathbf{I} \subseteq {}^m\mathfrak{C}$, I an infinite linear order, $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$, we say that $\bar{\mathbf{b}}$ is an (A, D) -indiscernible sequence if for each $t \in I$ we have $\text{tp}(\bar{b}_t, A \cup \{\bar{b}_s : s <_I t\}) = \text{Av}(A \cup \{\bar{b}_s : s <_I t\}, D)$. If $A = \cup\{\bar{c} : \bar{c} \in \text{Dom}(D)\}$, we may write “ $\bar{\mathbf{b}}$ is a D -indiscernible sequence”.

3) $\text{Av}_\varphi(A, D)$ where $\varphi = \varphi(\bar{x}, \bar{y})$ is $\{\varphi(\bar{x}, \bar{a}) : \varphi(\bar{x}, \bar{a}) \in \text{Av}(A, D)\}$ and $\text{Av}_\Delta(A, D) = \bigcup_{\varphi \in \Delta} \text{Av}_\varphi(A, D)$.

1.4 Claim. 1) For D an ultrafilter on $\mathbf{I} \subseteq {}^m\mathfrak{C}$ and $B \subseteq \mathfrak{C}$ we have $\text{Av}(B, D) \in S^m(B)$.

2) If $\mathbf{I} \subseteq {}^m A$ and D is an ultrafilter on \mathbf{I} and $\langle \bar{b}_t : t \in I \rangle$ is an (A, D) -indiscernible sequence then $\langle \bar{b}_t : t \in I \rangle$ is an indiscernible sequence over A .

Proof. Check.

1.5 Definition. 1) For an infinite linear order I and an indiscernible sequence $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$, having $\ell g(\bar{b}_t) = m$ for $t \in I$, we define:

- (a) $\text{avf}_{\text{pa}}(\bar{\mathbf{b}}) = \{\varphi(\bar{x}, \bar{y}, \bar{c}) : \ell g(\bar{y}) = m, \text{ and for every } \bar{a} \in {}^{\ell g(\bar{x})}\mathfrak{C}, \text{ the set } \{t \in I : \mathfrak{C} \models \varphi(\bar{a}, \bar{b}_t, \bar{c})\} \text{ is finite } \underline{\text{or}} \text{ the set } \{t \in I : \mathfrak{C} \models \neg\varphi(\bar{b}_t, \bar{a}, \bar{c})\} \text{ is finite}\}$
(avf stands for averagable formulas, pa stands for parameters)
- (b) $\text{avf}(\bar{\mathbf{b}}) = \{\varphi(\bar{x}, \bar{y}) : \varphi(\bar{x}, \bar{y}) \in \text{avf}_{\text{pa}}(\bar{\mathbf{b}}), \text{ i.e. no parameters}\}$
- (c) $\text{daf}(\bar{\mathbf{b}}) = \{\varphi(\bar{x}, \bar{y}) : \ell g(\bar{y}) = m \text{ and for every } \bar{a} \in {}^{\ell g(\bar{x})}\mathfrak{C} \text{ the set } \{t \in I : \mathfrak{C} \models \varphi[\bar{a}, \bar{b}_t]\} \text{ is a finite union of convex subsets of } I\}$.

Let $\text{daf}_{\text{pa}}(\bar{\mathbf{b}})$ be defined similarly allowing parameters,

- (d) $\text{daf}^n(\bar{\mathbf{b}})$ when the union is of $\leq n$ convex sets; similarly is the other cases.

2) For a sequence $\bar{\mathbf{b}} = \langle \bar{b}_t : t < k \rangle$ with $\bar{b}_\ell \in {}^m\mathfrak{C}$, and formula $\varphi = \varphi(\bar{y}, \bar{z})$, $\ell g(\bar{y}) = m$, we define

$$\begin{aligned} \text{av}_\varphi(A, \langle \bar{b}_\ell : \ell < k \rangle) &= \{\varphi(\bar{y}, \bar{c})^{\mathbf{t}} : \mathbf{t} \in \{\text{true}, \text{false}\}, \\ &\quad \bar{c} \in {}^{\ell g(\bar{z})}A, \text{ and } |\{\ell : \models \varphi(\bar{b}_\ell, \bar{c})^{\mathbf{t}}\}| > k/2\} \end{aligned}$$

(this is not necessarily a type, just a set of formulas).

3) $E = E_{\varphi(\bar{y}, \bar{z})}^k$, a formula in $L(\tau_T)$, written $\bar{z}_1 E \bar{z}_2$ with $\ell g(\bar{z}_1) = \ell g(\bar{z}_2) = (\ell g(\bar{x})) \times k$ (written $(\bar{x}_1, \dots, \bar{x}_n)$ instead of $\bar{x}_1 \wedge \dots \wedge \bar{x}_n$, abusing notation) is defined as follows:
 $(\bar{x}_0, \dots, \bar{x}_{k-1}) E (\bar{x}'_0, \dots, \bar{x}'_{k-1}) =:$

$$(\forall \bar{z}) \left(\bigvee_{u \subseteq k, |u| > k/2} \bigwedge_{\ell \in u} \varphi(\bar{x}_\ell, \bar{z}) \equiv \bigvee_{u \subseteq k, |u| > k/2} \bigwedge_{\ell \in u} \varphi(\bar{x}'_\ell, \bar{z}) \right).$$

Of course, it is an equivalence relation.

1.6 Claim. *If $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ is an infinite indiscernible sequence, $\ell g(\bar{\mathbf{b}}) = m$ and $\varphi(\bar{y}; \bar{z}) \in \text{avf}(\bar{\mathbf{b}})$ so $\ell g(\bar{z}) = m$, then for every k large enough we have:*

- (a) *for any \bar{c} of length $\ell g(\bar{z})$, for some truth value \mathbf{t} the set $\{t \in I : \models \varphi(\bar{b}_t, \bar{c})^{\mathbf{t}}\}$ has $< k/2$ members*
- (b) *if t_0, \dots, t_{k-1} are distinct members of I then $\text{av}_\varphi(\langle \bar{b}_{t_\ell} : \ell < k \rangle, \mathfrak{C}) \in S_\varphi^{\ell g(\bar{m})}(\mathfrak{C})$, in fact for every nonprincipal ultrafilter D over $\{t \in I\}$ and set A we have $\text{av}_\varphi(\langle \bar{b}_{t_\ell} : \ell < k \rangle, A)$ is a subset of $\text{Av}(A, D)$, in fact is $\text{Av}_\varphi(A, D)$*
- (c) *if $t_0, \dots, t_{k-1} \in I$ with no repetitions and $s_0, \dots, s_{k-1} \in I$ with no repetition then $(\bar{b}_{t_0}, \dots, \bar{b}_{t_{k-1}}) E_{\varphi(\bar{x}, \bar{y})}^k (\bar{b}_{s_0}, \dots, \bar{b}_{s_{k-1}})$*
- (d) *for some finite Δ : if $I', I \subseteq J$ where J is a linear order, $\bar{\mathbf{b}}' = \langle \bar{b}'_t : t \in J \rangle$, $\bar{\mathbf{b}}' \upharpoonright I = \bar{\mathbf{b}}$ and $\bar{\mathbf{b}}'$ is Δ -indiscernible sequence, $|I'| \geq k$, then*
 - (α) *(b), (c) holds for $\bar{\mathbf{b}}' \upharpoonright I'$ and*
 - (β) *$\varphi(\bar{y}, \bar{z}) \in \text{avf}(\bar{\mathbf{b}}' \upharpoonright I)$.*

Proof.

- (a) By compactness.
- (b), (c) Just think of the definitions.
- (d) By compactness. □_{1.6}

1.7 Definition. Let $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ an infinite indiscernible sequence.

1) We define (Cb stands for canonical bases):

- (a) for $\varphi(\bar{y}; \bar{z}) \in \text{avf}(\bar{\mathbf{b}})$ let $Cb_{\varphi(\bar{y}; \bar{z})}(\bar{\mathbf{b}})$ be $(\bar{b}_{t_0}, \dots, \bar{b}_{t_{k-1}}) / E_{\varphi(\bar{y}, \bar{z})}^k \in \mathfrak{C}^{\text{eq}}$, with $k = k_{\varphi(\bar{y}, \bar{z})}(\bar{\mathbf{b}})$ minimal as in 1.6(a)
- (b) $Cb(\bar{\mathbf{b}}) = \text{dcl}\{Cb_{\varphi(\bar{y}, \bar{z})}(\bar{\mathbf{b}}) : \varphi(\bar{y}, \bar{z}) \in \text{avf}(\bar{\mathbf{b}})\} \subseteq \mathfrak{C}^{\text{eq}}$.

2) If I has no last element then $\text{Av}(A, \bar{\mathbf{b}}) = \{\varphi(\bar{x}, \bar{a}) : \bar{a} \in {}^\omega A \text{ and } \models \varphi(\bar{b}_t, \bar{a}) \text{ for every large enough } t \in I\}$, $\text{Av}_\varphi(A, \bar{\mathbf{b}}) = \{\varphi(\bar{x}, \bar{a}) : \text{for every large enough } t \in I \text{ we have } \mathfrak{C} \models \varphi(\bar{b}_t, \bar{a})\}$.

3) Let $\text{Av}_{\text{avf}}(\mathbf{A}, \bar{\mathbf{b}})$ be $\text{Av}_\Delta(\bar{\mathbf{b}}, A)$ for $\Delta = \text{avf}(\bar{\mathbf{b}})$, similarly for replacements to avf .

1.8 Claim. *[T has NIP] Assume I is a linear order with no last element and $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ an indiscernible sequence, $\text{lg}(\bar{\mathbf{b}}) = n$.*

- 1) $\text{av}_\varphi(A, \bar{\mathbf{b}}) \in S_\varphi^m(A)$, see Definition 1.5(a).
- 2) $\text{Av}(A, \bar{\mathbf{b}}) \in S^m(A)$, see Definition 1.7(c).

Proof. By [Sh:c, II.4.13].

To formalize clause (d) of 1.6 let

1.9 Definition. 1) For a set Δ of formulas and $k \leq \omega$ we say that $\langle \bar{b}_t^1 : t \in I_1 \rangle, \langle \bar{b}_t^2 : t \in I_2 \rangle$ are immediate (Δ, k) -nb-s if:

- (a) both are Δ -indiscernible sequences of length $\geq k$
- (b) for some Δ -indiscernible sequence $\langle \bar{b}_t : t \in I \rangle$ we have $I_\ell \subseteq I, (\forall t \in I_\ell) \bar{b}_t^\ell = \bar{b}_t$ for $\ell = 1, 2$.

2) The relation “being (Δ, k) -nb-s” is the closure of being an “immediate (Δ, k) -nb” to an equivalence relation. We say “of distance k ” if there is a chain of immediate (Δ, k) -nb-s of length k of length $\leq k$ starting with one ending in the other. We write Δ instead of (Δ, ω) if $\Delta = L(T)$ we may omit Δ .

3) If $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are infinite indiscernible sequences, we say there are “essentially nb-s” if for every finite $\delta \in L(T), k < \omega$ they are (Δ, k) -nb-s.

4) If $\bar{\mathbf{b}}$ is an infinite indiscernible sequence over A we let $C_A(\bar{\mathbf{b}}) = \{\bar{b}_t : \text{for some } \bar{\mathbf{b}}' \text{ essentially nb-s of } \bar{\mathbf{b}}, \bar{b} \text{ appears in } \bar{\mathbf{b}}'\}$.

1.10 Claim. 1) If $\langle \bar{b}_t : t \in I \rangle$ is an infinite indiscernible sequence and $\varphi(\bar{y}, \bar{z}) \in \text{daf}^n(\langle \bar{b}_t : t \in I \rangle)$, then for some finite Δ and k , for any (Δ, k) -nb-s $\langle \bar{b}_t' : t \in I' \rangle$ of $\langle \bar{b}_t : t \in I \rangle$ we have $\varphi(\bar{y}, \bar{z}) \in \text{daf}^n(\langle \bar{b}_t' : t \in I' \rangle)$.

2) The result in (1) holds also for $\text{avf}^n(\langle \bar{b}_t : t \in I \rangle)$. If $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ is an infinite indiscernible sequence and $\varphi(\bar{y}, \bar{z}) \in \text{avf}(\bar{\mathbf{b}})$, then for some finite Δ and k for any (Δ, k) -nb $\bar{\mathbf{b}}'$ of $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ we have $\text{Cb}_{\varphi(\bar{y}, \bar{z})}(\bar{\mathbf{b}}') = \text{Cb}_{\varphi(\bar{y}, \bar{z})}(\bar{\mathbf{b}})$.

3) If T has NIP and $\bar{\mathbf{b}}$ is an infinite indiscernible sequence then every $\varphi(\bar{y}, \bar{z})$ with the right length of \bar{y} belongs to $\text{daf}(\bar{\mathbf{b}})$ (see [Sh:c, Ch.II, §4]).

4) If \bar{b}_1, \bar{b}_2 are (Δ, k) -nb-s of distance n for every finite Δ and $k < \omega$, then they are $(L(T), k)$ -nb-s of distance n .

Proof. Easy.

* * *

1.11 Definition/Claim. Assume that $M \prec \mathfrak{C}, p \in S^m(M)$ and for $\ell = 1, 2$, D_ℓ is an ultrafilter on ${}^m M$ and $\bar{\mathbf{b}}^\ell = \langle \bar{b}_t^\ell : t \in I_\ell \rangle$ is an infinite D_ℓ -indiscernible sequence over M such that $\text{Av}(M, D_\ell) = p$ (so does not depend on ℓ). Then

- (a) for every finite set Δ_1 of formulas, finite $A \subseteq M$ and $j \in \{1, 2\}$ there is a function \mathcal{F} into D_j such that
 - (*) if $\alpha \leq \omega$ and for each $\ell < \alpha$ we have $\bar{b}_\ell \in {}^m M, \bar{b}_\ell \in \mathcal{F}(\bar{b}_0, \dots, \bar{b}_{\ell-1}) (\in D_j)$ then the sequence $\langle \bar{b}_t^j : t \in I \rangle \wedge \langle \bar{b}_\ell : \ell \in \alpha^* \rangle$ (where the $*$ in α^* means invert the order) is Δ_1 -indiscernible sequence over A
- (b) for any $\varphi(\bar{y}, \bar{z})$ for finite large enough $\Delta \subseteq L(T)$, if $j \in \{1, 2\}$ and $\bar{b}_\ell \in {}^m M$ for $\ell < k$ where $k < \omega$ is large enough are as in clause (a), we have: if $j \in \{1, 2\}$ and $\varphi = \varphi(\bar{y}, \bar{z}) \in \text{avf}(\bar{\mathbf{b}}^j)$, k also large enough as in 1.6 then
 - (i) $\text{av}_\varphi(\mathfrak{C}, \bar{\mathbf{b}}^j) = \text{av}_\varphi(\mathfrak{C}, \langle \bar{b}_\ell : \ell < k \rangle)$
 - (ii) $p \upharpoonright \varphi(\bar{y}, \bar{z})$ is definable as $\{ \varphi(y, \bar{c})^t : \bar{c} \subseteq M \text{ and } |\{ \ell < k : \neg \varphi(\bar{b}_\ell, \bar{c})^t \}| < k/2 \}$, so a first order formula with parameters from M
 - (iii) $\text{Av}_{\varphi(\bar{y}, \bar{z})}(\mathfrak{C}, \bar{\mathbf{b}}^j)$ does not depend on j if $\varphi(\bar{y}, \bar{z}) \in \text{avf}(\bar{\mathbf{b}}^1) \cap \text{avf}(\bar{\mathbf{b}}^2)$
- (c) $\text{avf}(\bar{\mathbf{b}}^1) = \text{avf}(\bar{\mathbf{b}}^2)$ so we can call it $\text{avf}(p)$!
- (d) similarly $\text{Cb}_\varphi(\bar{\mathbf{b}}^1) = \text{Cb}_\varphi(\bar{\mathbf{b}}^2)$ call it $\text{Cb}_\varphi(p)$ and $\text{Cb}(\bar{\mathbf{b}}_1) = \text{Cb}(\bar{\mathbf{b}}_2)$ call it $\text{Cb}(p)$, so $p \upharpoonright \text{avf}(p)$ is definable with parameters from $\text{Cb}(p)$
- (e) if $\varphi(\bar{y}, \bar{z}) \in \text{avf}(p)$, then

$$\text{Av}_{\varphi(\bar{y}, \bar{z})}(\mathfrak{C}, \bar{\mathbf{b}}^1) = \text{Av}_{\varphi(\mathfrak{C}, \bar{y}, \bar{z})}(\mathfrak{C}, \bar{\mathbf{b}}^2).$$

Proof. Straight.

Clause (a):

There is no harm with increasing I_j , so without loss of generality, (if we supply appropriate \bar{b}_t 's) I_j has no last element. We then prove by induction on α (for all A).

Case 1: $\alpha = 0$ Nothing to prove.

Case 2: $\alpha = 1$

Let $n < \omega$ be above the number of free variables in any formula in Δ_1 and let $t_0 < \dots < t_{n-1} < t$ be in I_j . Now $\text{tp}_{\Delta_1}(\bar{b}_t, A \cup \bigcup_{\ell < n} \bar{b}_{t_\ell}, \mathfrak{C})$ is a finite set of formulas

and let $\psi(\bar{x}, \bar{c})$ be its conjunction. So $\psi(\bar{x}, \bar{c}) \in \text{tp}(\bar{b}_t, M \cup \{\bar{b}_s : s < t\}, \mathfrak{C}) = \text{Av}(M \cup \{\bar{b}_s : s < t\}, D_j)$ hence $\mathbf{I} = \{\bar{b} \in {}^m M : \mathfrak{C} \models \psi(\bar{b}, \bar{c})\} \in D_j$.

So define $\mathcal{F} = \mathcal{F}_A$ by $\mathcal{F}() = \mathbf{I}$, why is it as required? Clearly $\bar{b}' \in \mathcal{F}()$ implies that $[t_{n-1} < t' \in I_j \Rightarrow \bar{b}_{t'}, \bar{b}'$ realizes the same Δ_1 -type over $A \cup \{\bar{b}_t, \dots, \bar{b}_{t_{n-1}}\}$.

But as $\langle \bar{b}_s : s \in I_j \rangle$ is an indiscernible sequence over M , and $\bar{b}' \in M$, we can replace $t_0 < \dots < t_{n-1}$, by any $t'_0 < \dots < t'_{n-1}$.

But by the choice of n , for any \bar{b} we have

$$\text{tp}_{\Delta_1}(\bar{b}, A \cup \{\bar{b}_s : s \in I\}) = \cup \{ \text{tp}_{\Delta_1}(\bar{b}, A \cup \{\bar{b}_{s_0}, \dots, \bar{b}_{s_{n-1}}\}) : s_0 < \dots < s_{n-1} \text{ are in } I_j \}$$

so we are done.

Case 3: $\alpha > 1, \alpha < \omega$.

We define $\mathcal{F}(\bar{b}_0, \dots, \bar{b}_{\ell-1})$ as $\mathcal{F}_{A \cup \{\bar{b}_0, \dots, \bar{b}_{\ell-1}\}}$, and the checking is easy.

Clause (b):

Subclause (i) holds by Claim 1.6, Clause (d).

Subclause (ii) follows from subclause (i) as

$$\begin{aligned} \boxtimes \text{ for } \bar{c} \in {}^{(\ell g(\bar{z}))} M \text{ we have: } \varphi(x, \bar{c}) \in p \text{ iff } \varphi(\bar{x}, \bar{c}) \in \text{Av}(M, \bar{\mathbf{b}}^j) \text{ iff} \\ \varphi(\bar{y}, \bar{c}) \in \text{av}_{\varphi}(\mathfrak{C}, \langle \bar{b}_{\ell} : \ell < k \rangle) \text{ iff } \mathfrak{C} \models \vartheta[\bar{c}, b_0, \dots, b_{k-1}] \text{ where} \\ \vartheta(y, b_0, \dots, \bar{b}_{k-1}) = \bigvee_{u \subseteq k, |u| \geq k/r} \bigvee_{\ell \in u} \varphi(y, \bar{b}_{\ell}) \end{aligned}$$

[why? the first “iff” as $\text{Av}_{\varphi}(\mathfrak{C}, \bar{\mathbf{b}}^j)$ restricted to M is $p \upharpoonright \varphi$, by an assumption of our claim, the second iff by clause (a) which we have proved, the third iff by Definition 1.5(2).]

To Subclause (iii), we can choose for $j = 1, 2$, sequence $\bar{b}_{\ell}^j \in {}^m M$ for $\ell < k$ as in clause (a), so for any $\bar{c} \in {}^{(\ell g(\bar{z}))} \mathfrak{C}$ we have

$$(*) \quad \varphi(\bar{y}, \bar{c}) \in \text{Av}_{\varphi(\bar{y}, \bar{z})}(\mathfrak{C}, \bar{\mathbf{b}}^j) \text{ iff } \mathfrak{C} \models \vartheta[\bar{c}, \bar{b}_0^j, \dots, \bar{b}_{k-1}^j].$$

So if the conclusion fails then for some \bar{c} we have $\mathfrak{C} \models \Theta[\bar{c}, \bar{b}_0^1, \dots, \bar{b}_{k-1}^1] = \neg \Theta[\bar{c}, \bar{b}_0^2, \dots, \bar{b}_{k-1}^2]$. As $\bar{b}_{\ell}^j \in {}^m M$, $M \prec \mathfrak{C}$ clearly there is $\bar{c}' \subseteq M$ as above but by subclause (2)

$$M \models \Theta[\bar{c}', b_0^j, \dots, b_{k-1}^j] \Leftrightarrow \varphi(\bar{y}, \bar{c}') \in p.$$

We get contradiction.

Clause (c):

This follows by 1.6 clause (d) and 1.11, clause (a) above.

Clause (d):

Similar to the proof of clause (c), using 1.10(2).

Clause (e):

This is (b)(iii) + (c).

□_{1.11}

Of course

1.12 Observation. For any $M \prec \mathfrak{C}$ and $p \in S^m(M)$, $\text{avf}(p)$, $Cb(p)$ are well defined as there are ultrafilters D on ${}^m M$ such that $\text{Av}(M, D) = p$.

1.13 Observation [T has NIP] 1) If $\bar{\mathbf{b}}$ is an infinite indiscernible set over \emptyset (i.e. order immaterial), then

(a) $\text{avf}(\bar{\mathbf{b}}) = L(T)$ (i.e. all formulas)

(b) if $\bar{\mathbf{b}}$ is an indiscernible sequence over A then it is an indiscernible set over A ;

(b)⁺ if $\bar{\mathbf{b}}$ is a $\{\varphi(\bar{x}_0, \dots, \bar{x}_k; \bar{c})\}$ -indiscernible sequence but is a Δ_φ -indiscernible set (over \emptyset), then $\bar{\mathbf{b}}$ is a $\{\varphi(\bar{x}_0, \dots, \bar{x}_{k-1}; \bar{c})\}$ -indiscernible set when $\Delta_\varphi = \{\exists \bar{x} \bigwedge_{\ell < n} \varphi(\bar{x}, \bar{y}_\ell)^{\text{if} \eta(\ell)} : \eta \in {}^n 2\}$ and n is such that $\boxtimes_{\varphi(\bar{x}, \bar{y})}^n$ from 1.2 fail.

2) If $p \in S^m(M)$ and $D_j, \bar{\mathbf{b}}^j$ for $j = 1, 2$ are as in 1.11, then $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are nb-s of distance 2.

Proof. 1)(a) By [Sh:c, Ch.II, 4.13] + the definitions.

(b) Easy by clause (a), see details in 4.6(1).

(b)⁺ Similarly.

2) By compactness and the proof of 1.11.

□_{1.13}

1.14 Conclusion. [T has NIP] In 1.11, $\bar{\mathbf{b}}^1$ is an indiscernible set over \emptyset iff $\bar{\mathbf{b}}^1$ is an indiscernible set over M iff $\bar{\mathbf{b}}^2$ is an indiscernible set over \emptyset .

1.15 Definition. If $p \in S(M)$ and for some (\equiv all) $D^1, \bar{\mathbf{b}}^1$ as in 1.11 we have $\bar{\mathbf{b}}^1$ is an indiscernible set, then we call p a stable type. Otherwise p is called nonstable type.

1.16 Conclusion. $[T$ has NIP] 1) If $p \in S^m(M)$ is a stable type, then each $p \upharpoonright \varphi$ is definable, in fact by parameters from $Cb(p)$.

2) The number of stable $p \in S^m(M)$ is $\leq \|M\|^{|T|}$.

Proof. 1) By 1.11 (use clause (a) in 1.11).

2) Count the possible number of Definitions. $\square_{1.16}$

1.17 Remark. Note that $p \in S(M)$ may be definable but not stable, e.g. $M \prec N$ are models of the theory of $(\mathbb{R}, <)$, and $a \in N \setminus M$ is above all $b \in M$, then $\text{tp}(a, M, N)$ is definable but not stable.

1.18 Observation: $[T$ is NIP and unstable].

1) There are $M \prec \mathfrak{C}$ and nonstable $p \in S^1(M)$ [in \mathfrak{C} not \mathfrak{C}^{eq}].

2) There is an indiscernible sequence of elements which is not an indiscernible set of elements (over \emptyset !)

3) If $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ is an indiscernible sequence of m -tuples but not an indiscernible set, say for $\varphi(\bar{x}_0, \dots, \bar{x}_{k-1})$, I a dense (linear order) for simplicity then for some $i < k-1$ and some $t_0, \dots, t_{k-1} \in I$ with no repetitions such that $t_i < t_{i+1}$, $(t_i, t_{i+1})_I \cap \{t_0, \dots, t_{k-1}\} = \emptyset$ we have: the formula $\psi(\bar{b}_{t_0}, \dots, \bar{b}_{t_{i-1}}, \bar{x}_i, x_{i+1}, \bar{b}_{t_{i+1}}, \dots)$ define a partial order on ${}^m\mathfrak{C}$ by which $\langle \bar{b}_t : t \in I, t_i <_I t <_I t_{i+1} \rangle$ is strictly increasing where

$$\psi(\bar{x}_{t_0}, \dots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_{k-1}) = \forall \bar{y} [\varphi(\bar{x}_{t_0}, \dots, \bar{x}_{i-1}, \bar{x}_i, \bar{y}, x_{i+2}, \dots, \bar{x}_{k-1}) \rightarrow \varphi(\bar{x}_{t_0}, \dots, \bar{x}_{i-1}, \bar{x}_{i+2}, \dots, \bar{x}_{i+2}, \dots, \bar{x}_{k-1})].$$

Proof. 1) As T is unstable, for some $M \prec \mathfrak{C}$ we have $|S(M)| > \|M\|^{|T|}$ hence 1.16(2) some type $p \in S(M)$ is nonstable.

2) By part 1) and Definition 1.15.

3) Well known, see [Sh:c, Ch.II, §4] $\square_{1.18}$

Now 1.18(3) applies to 1.18(2) (where \bar{x}_i is x_i) gives

1.19 Conclusion. If T is (NIP but) unstable, then some formula $\varphi(x, y; \bar{c})$ define on \mathfrak{C} a quasi order with infinite chains, (so x, y singletons).

1.20 Remark. So if T satisfies some version of $*$ -stable (see [Sh 300, Ch.II] or [Sh 702]) then T is stable or T has IP.

So we may wonder

Question: Does the “has NIP” case in 1.20 is needed? If T has IP does some $\varphi(x, y, \bar{c})$ have the independence property?

Note that

1.21 Claim. : *If T is unstable, then some formula $\varphi(x, y, \bar{c})$ has the order property (equivalently is unstable, hence some $\varphi(x(y, \bar{c}))$ define a partial order with infinite claims or has the independence property.*

Proof. We know that some $\varphi(x, \bar{y})$ is unstable so choose a formula $\varphi(x, \bar{y}, \bar{c})$ with the order property, such that $\ell g(\bar{y})$ is minimal. So there is an indiscernible sequence $\langle \bar{a}_i : i < \omega 4 \rangle$ such that $\mathfrak{C} \models \varphi[b_i, \bar{a}_j]$ iff $j < i$. Clearly $\langle b_i : i < \omega 4 \rangle$ is an indiscernible sequence over \bar{c} , if it is not an indiscernible set, say not (ϑ, k) -indiscernible set, $\vartheta = \vartheta(x_0, \dots, x_{k-1}, \bar{c})$, then for some permutation π of $\{0, \dots, k-1\}$ and m the formula $\vartheta(x, y, \bar{a}_0, \dots, a_{m-1}, a_{n-2}, a_{2\omega+1}, \dots, a_{2\omega+k, m-3}, \bar{c})$ linear orders $\langle a_{\omega+i} : i < \omega \rangle$, hence has the order property. So assume $\langle b_i : i < \omega \rangle$ is an indiscernible set over \bar{c} , and let a'_i be the first element of the sequence \bar{a}_i . If $\langle b_{2i+1} : i < \omega 4 \rangle$ is not an indiscernible sequence over $\bar{c} \cup \{a'_{2\omega}\}$ then we can find a formula $\vartheta(x, y, \bar{c}')$, $\bar{c}' \subseteq \bar{c} \cup \{a_i : i < \omega \text{ or } \omega 3 \leq i\}$ such that $\models \vartheta[b_{2\omega}, a_{\omega+2i+1} \bar{c}']$ for $i < \omega$ but $\models \neg \vartheta[b_{2\omega}, a_{\omega 2+2i+1} \bar{c}']$ for $i < \omega$ and we are done. So assume $\langle b_{2i+1} : i < \omega 4 \rangle$ is an indiscernible sequence over $\bar{c} \cup \{a'_{2\omega}\}$, hence all $\{a'_{2j} : j < \omega 4\}$ realizes the same type over $\{b_{2i+1} : i < \omega 4\} \cup \bar{c}$ hence for $j < 2\omega$ we can find \bar{a}_{2j}^* realizing $\text{tp}(\bar{a}_{2j}, \{b_{2i+1} : i < \omega 4\} \cup \bar{c}, \mathfrak{C})$ and the first element of \bar{a}_{2j}^* is a'_0 . This contradicts the choice of $\varphi(x, \bar{y}, \bar{c})$ as having the order property with $\ell g(\bar{y})$ minimal as we can “move” a'_0 to \bar{c} . $\square_{1.21}$

* * *

Remark. Note that for indiscernible sets, the theorems on dimension in [Sh:c, III] holds for theories T with NIP, see §3.

§2 CHARACTERISTICS OF TYPES

We continue to speak on canonical bases and we deal with the characteristics of types and of indiscernible sets. More elaborately, for any indiscernible sequence $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$, I an infinite linear order, we have a measure $\text{ch}(\bar{\mathbf{b}}) = \langle CH_{\varphi(\bar{x}, \bar{y}_\varphi)}(\bar{\mathbf{b}}) : \varphi(\bar{x}, \bar{y}_\varphi) \in L(T) \rangle$ with $\bar{\lambda} = \langle x_i : i < m \rangle$, $m = \ell g(\bar{b}_t)$ for $t \in J$, where $\text{Ch}_{\varphi(\bar{x}, \bar{y}_\varphi)}(\bar{\mathbf{b}})$ measure how badly $\varphi(\bar{x}, \bar{y}_\varphi)$ fail to be in $\text{avf}(\bar{\mathbf{b}})$ (see Definition 2.6), we can find such $\bar{\mathbf{b}}$'s with maximal such $\text{CH}(\bar{\mathbf{b}})$ and wonder what can we say about them.

2.1 Hypothesis. T is NIP.

2.2 Definition/Claim. Let $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ be an infinite indiscernible sequence, $k < \omega$. Then

- (a) (Claim) if $t_i \in I$ and $i < j \Rightarrow t_i <_I t_j$ for $i < j < \omega$ and $\bar{\mathbf{b}}^k = \langle \bar{b}_{t_{ki}} \wedge \bar{b}_{t_{ki+1}} \wedge \dots \wedge \bar{b}_{t_{ki+k-1}} : i < \omega \rangle$ then
 - (α) $Cb(\bar{\mathbf{b}}^1) \subseteq Cb(\bar{\mathbf{b}}^k)$,
 - (β) if $\varphi'(\bar{x}_1, \dots, \bar{x}_k; \bar{y}) = \varphi(\bar{x}_\ell, \bar{y})$ then:
 $\varphi'(\bar{x}_1, \dots, \bar{x}_k; \bar{y}) \in \text{avf}(\bar{\mathbf{b}}^k)$ iff $\varphi(\bar{x}; \bar{y}) \in \text{avf}(\bar{\mathbf{b}})$
 - (γ) if $\bar{\mathbf{b}}^{k,1}, \bar{\mathbf{b}}^{k,2}$ are related like $\bar{\mathbf{b}}^k$ above to our $\bar{\mathbf{b}}$ then $Cb(\bar{\mathbf{b}}^{k,1}) = Cb(\bar{\mathbf{b}}^{k,2})$
 - (δ) if $\varphi(\bar{x}, \bar{y}) \in \text{dof}(\bar{\mathbf{b}})$, $\varphi' = \varphi'(\bar{x}_1, \dots, \bar{x}_k, \bar{y}) = \varphi(\bar{x}_\ell, \bar{y})$ & $\neg \varphi(\bar{x}_m, \bar{y})$ or $= \varphi(\bar{x}_\ell, \bar{y}) \equiv \neg \varphi(\bar{x}_m, \bar{y})$ then $\varphi' \in \text{avf}(\bar{\mathbf{b}})^k$
- (b) (Definition) let $Cb^k(\bar{\mathbf{b}}) = Cb(\bar{\mathbf{b}}^k)$, $\text{Av}^k(\bar{\mathbf{b}}, \mathfrak{C}) = \text{Av}_{\text{avf}}(\bar{\mathbf{b}}^k, \mathfrak{C})$ for any $\bar{\mathbf{b}}^k$ as above
- (c) (Definition) $Cb^\omega(\bar{\mathbf{b}}) = \cup \{Cb^k(\bar{\mathbf{b}}) : k < \omega\}$
- (d) (Fact) if I_1, I_2 are infinite subsets of J and $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in J \rangle$ an indiscernible sequence (recall J linear order) then
 $Cb^\omega(\bar{\mathbf{b}} \upharpoonright I_1) = Cb^\omega(\bar{\mathbf{b}} \upharpoonright I_2)$.
- (e) (Fact) If the infinite indiscernible sequences $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are nb-s, then $Cb^\alpha(\bar{\mathbf{b}}^1) = Cb^\alpha(\bar{\mathbf{b}}^2)$ for $\alpha \leq \omega$.

Proof. Easy.

2.3 Definition. For $\alpha \leq \omega$. We say $p \in S^m(A)$ does not α -fork over $B \subseteq A$, if for some model $M \supseteq A$ and $q \in S^m(M)$ extending p we have $Cb^\alpha(q) \subseteq acl_{\mathfrak{C}^{eq}}(B)$. Similarly we say that C/B does not α -fork over $A \subseteq B$ if $\bar{c} \subseteq C \Rightarrow \text{tp}(C, B)$ does not α -fork over it.

2.4 Claim. 1) In 2.3: “for some $M \supseteq A$ ” can be replaced by “for every $M \supseteq A$ ”.

Proof. Easy.

2.5 Remark. : Assume that T is a simple theory, $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ is an infinite indiscernible sequence. Then we cannot find $\langle \bar{a}_n : n < \omega \rangle$ indiscernible, $\langle \varphi(\bar{x}, \bar{a}_n) : n < \omega \rangle$ pairwise contradictory (or just m -contradiction for some m) and

$$\bigwedge_{n < \omega} (\exists^\infty t \in I)(\varphi(\bar{b}_t, \bar{a}_n).$$

Proof. As we can repeat and get the tree property. More fully, for any cardinals $\mu > \kappa$ we consider $J = {}^\kappa\mu$ as a linearly ordered set, ordered lexicographically and for $\rho \in {}^\kappa\mu$ let $J_\rho = \{\nu \in J : \rho \triangleleft \nu\}$; without loss of generality I is countable and $h : I \rightarrow J$ is order preserving. We can find $\bar{a}_\eta \in \mathfrak{C}$ for $\eta \in J$ such that $\langle c_\eta : \eta \in J \rangle$ is an indiscernible sequence such that $t \in I \Rightarrow c_{h(t)} = b_t$. By compactness, for each $\alpha < \kappa$ we can find $\langle a_\rho : \rho \in {}^\alpha\mu \rangle$ such that:

- (α) $\langle \varphi(\bar{x}, \bar{a}_\rho) : \rho \in {}^\alpha\mu \rangle$ are pairwise contradictory (or just any m of them)
- (β) $\eta \in J_\rho, \rho \in {}^\alpha\mu \Rightarrow \mathfrak{C} \models \varphi[c_\eta, a_\rho]$.

Now $\langle \varphi(\bar{x}, \bar{a}_\rho) : \rho \in {}^\kappa\mu \rangle$ exemplified the tree property. $\square_{2.5}$

We have looked at indiscernible sequences which are stable. We now look after indiscernible sequences which are in the other extreme.

2.6 Definition. For $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ an indiscernible sequence, we define its character

$$Ch(\bar{\mathbf{b}}) = \langle Ch_{\varphi(\bar{y}, \bar{z})}(\bar{\mathbf{b}}) : \varphi(\bar{y}, \bar{z}) \in L(T) \rangle$$

where

$$Ch_{\varphi(\bar{y}, \bar{z})}(\bar{\mathbf{b}}) = \text{Max}\{n : \text{for some } \bar{c}, \langle \text{TV}(\varphi(\bar{b}_t, \bar{c}) : t \in I) \rangle \text{ change sign } n \text{ times (i.e. } I \text{ divided to } n + 1 \text{ intervals)}\}.$$

2) For $p \in S^m(A)$, let

- (a) $CH(p) = \{Ch(\bar{\mathbf{b}}) : \bar{\mathbf{b}} \text{ is an infinite indiscernible sequence such that every } \bar{b}_t \text{ realizes } p\}$
- (b) for a formula $\varphi = \varphi(\bar{x}_0, \dots, \bar{x}_{k-1})$ let $CH(p, \varphi(\bar{x}_0, \dots, \bar{x}_k)) = \{Ch(\bar{\mathbf{b}}) : \bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle \text{ is an infinite indiscernible set such that } t_0 <_I t_1 <_I \dots <_I t_{k-1} \Rightarrow \mathfrak{C} \models \varphi[\bar{b}_{t_0}, \dots, \bar{b}_{t_{k-1}}]\}$
- (c) $CH^{\max}(p) = \{\bar{n} \in CH(p) : \text{there is no bigger such } \bar{n}' \in CH(p)\}$, when “ \bar{n}' is bigger than \bar{n} ” mean $(\forall \varphi)(n_\varphi \leq n'_\varphi) \ \& \ (\exists \varphi)(n_\varphi < n'_\varphi)$
- (d) $CH^{\min}(p, \varphi(\bar{x}_0, \dots, \bar{x}_{k-1})) = \{\bar{n} \in CH(p, \varphi(\bar{x}_0, \dots, \bar{x}_{k-1})) : \text{there is no smaller } \bar{n}' \in CH(p, \varphi(\bar{x}_0, \dots, \bar{x}_{k-1}))\}$.

Note: for the trivial φ , $CH(p, \varphi) = CH(p)$ hence $CH^{\max}(p, \varphi) = CH^{\max}(p)$.

2.7 Claim. Let $p \in S^m(A)$ be non-algebraic, $\bar{x} = \langle x_\ell : \ell < n \rangle$.

- 1) If $\bar{n} = \langle n_\varphi : \varphi = \varphi(\bar{x}, \bar{y}) \rangle \in CH(p)$, then there is $\bar{n}' \in CH^{\max}(p)$ such that $\bar{n} \leq \bar{n}'$.
- 2) $CH^{\max}(p)$ is non-empty.
- 3) If $CH(p, \varphi) \neq \emptyset$ then $CH^{\min}(p, \varphi) \neq \emptyset$ and $CH^{\max}(p, \varphi) \neq \emptyset$.

Proof. Let $R, <$ be an n -place and $2n$ -place predicate not in τ_T and let

$$\begin{aligned} \Gamma_p = & Th(\mathfrak{C}_T, c)_{c \in A} \cup \{(\forall \bar{x})[R(\bar{x}) \rightarrow \Theta(\bar{x}, \bar{c})] : \Theta(\bar{x}, \bar{c}) \in p\} \\ & \cup \{(\exists \bar{x}_0, \dots, \bar{x}_{n-1})(\bigwedge_{\ell < k} R(\bar{x}_\ell) \ \& \ \bigwedge_{\ell < m} \bar{x}_\ell \neq \bar{x}_m) : n < \omega\} \\ & \cup \{\bar{x} < \bar{y} \rightarrow R(\bar{x}) \wedge R(\bar{y})\} \\ & \cup \{\text{“and } \leq \text{ linearly ordered } \{\bar{x} : R(\bar{x})\}”\} \\ & \cup \{(\forall \bar{x}_1), \dots, (\forall \bar{x}_m)(\forall \bar{y}_1) \dots (\forall \bar{y}_m)(\bar{x}_1 < \bar{x}_2 < \dots < \bar{x}_m \\ & \quad \& \ \bar{y}_1 < \dots < \bar{y}_m \rightarrow \psi(\bar{x}_1, \dots, \bar{x}_m, \bar{c}) \equiv \psi(\bar{y}_1, \dots, \bar{y}_m, \bar{c}) : \\ & \quad \bar{c} \subseteq A \text{ and } \psi \in L(\tau_T) \text{ and } m < \omega\} \end{aligned}$$

(with $\bar{x}_i = \langle x_{i,\ell} : \ell < m \rangle$). If $\lambda = |T| + \aleph_1$ we may omit it. For $\bar{n} = \langle n_{\varphi(\bar{x}, \bar{y})} : \varphi(\bar{x}, \bar{y}) \in L(\tau_T) \rangle$ and $\bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}_i) : i < |T| \rangle$ listing those formulas let

$$\begin{aligned} \Gamma_{\bar{n}, \bar{\varphi}} = & \{\vartheta_{n_i, \varphi_i} : i < |T|\} \text{ where} \\ \vartheta_{n, \varphi(\bar{x}, \bar{y})} = & (\exists \bar{y})(\exists \bar{x}_0, \dots, \exists \bar{x}_n)[\bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_n \ \& \\ & \bigwedge_{\ell < n} (\varphi(\bar{x}_\ell, \bar{y})) \equiv \neg \varphi(\bar{x}_{\ell+1}, \bar{y})]. \end{aligned}$$

Now easily

- (a) Γ_p is a consistent type (using p being non algebraic and Ramsey theorem)
- (b) $\Gamma_p \cup \Gamma_{\bar{n}, \bar{\varphi}}$ is consistent iff $\bar{n} \in \text{CH}(p)$
- (c) if $\bar{n} \leq \bar{n}'$ then $\Gamma_{\bar{n}, \bar{\varphi}} \subseteq \Gamma_{\bar{n}', \bar{\varphi}}$
- (d) if J is a directed order $\bar{n}_t = \langle n_{t, \varphi(\bar{x}, \bar{y}_\varphi)} : \varphi(\bar{x}, \bar{y}_\varphi) \rangle \in \text{CH}(p, \varphi)$ increases with $t \in J$ and $\bar{n}^* = \langle n_{\varphi(\bar{x}, \bar{y}_\varphi)}^* : \varphi(\bar{x}, \bar{y}_\varphi) \rangle, n^* = \max\{n_{t, \varphi(\bar{x}, \bar{y}_\varphi)} : t \in J\}$, then $n^* \in \text{CH}(\bar{p}^*, \varphi)$
- (e) like (d) investing the order.

Together we can deduce the desired conclusions. $\square_{2.7}$

2.8 Question: For $p \in S(A)$ (or just $p \in S(M)$) does indiscernible sequences $\bar{\mathbf{b}}$ of elements realizing p such that $\text{Ch}(\bar{\mathbf{b}}) \in \text{CH}^{\max}(p)$, $\text{Ch}(\bar{b}) \in \text{CH}^{\min}(p, \varphi)$ play a special role?

§3 SHRINKING INDISCERNIBLES

The case of indiscernible sets is easier so we ignore it.

3.1 Claim. *If $\bar{\mathbf{b}} = \langle \bar{\mathbf{b}}_t : t \in I \rangle$ is an indiscernible sequence over $A, \bar{c} \in \mathfrak{C}$ (so finite), then*

- (a) *there are $J \subseteq I, J^* \subseteq J, |J^*| \leq |T|$ such that*
 - (*) *if $n < \omega, \bar{s}, \bar{t} \in {}^n I, \bar{s} \sim_{J^*} \bar{t}$ (i.e. \bar{s}, \bar{t} realize the same quantifier free type in the linear order J) then $\bar{a}_{\bar{s}} = \langle \bar{a}_{s_\ell} : \ell < n \rangle, \bar{a}_{\bar{t}} = \langle \bar{a}_{t_\ell} : \ell < n \rangle$ realize in \mathfrak{C} the same type over $A \cup \bar{c}$*
- (b) *if we fix n and deal with φ -types we can demand $|J^*| < k_{\varphi, n} < \omega$*
- (c) *if in addition $\bar{\mathbf{b}}$ is an indiscernible set, then in (*) of clause (a) we can weaken $\bar{s} \sim_{J^*} \bar{t}$ to $(\forall \ell, k)[(s_\ell < s_k \equiv t_\ell < t_k) \ \& \ s_\ell \in J^* \equiv t_\ell \in J^* \rightarrow s_\ell = t_\ell]$.*

Proof.

- (a) by (b)
- (b) follows by Claim 3.3 below
- (c) similarly. □_{3.1}

* * *

3.2 Definition. 1) For a linear order $I, m^* \leq \omega, n \leq \omega, \alpha_\ell$ an ordinal, a model M and a set $A \subseteq M$, we say that $\bar{\mathbf{a}} = \langle a_{u, \alpha, \ell} : \ell < n, u \in [I]^\ell, \alpha < \alpha_{|u|} \rangle$ is (Δ^*, m^*) -indiscernible over A of the $\langle \alpha_\ell : \ell < n \rangle$ -kind if the following holds:

- (*) if $m \leq m^*, I \models t_0 < \dots < t_{m-1}, I \models s_0 < \dots < s_{m-1}$ for $v \subseteq m$ we let $u_v = \{t_\ell : \ell \in v\}, w_v = \{s_\ell : \ell \in v\}$ then $\langle a_{u_v, \alpha, \ell} : \text{for } \ell \leq m, \ell < n, v \in [m]^\ell, \alpha < \alpha_\ell \rangle$ and $\langle a_{w_v, \alpha, \ell} : \ell \leq m, v \in [m]^\ell, \alpha < \alpha_\ell \rangle$ realizes the same (Δ, m^*) -type over A in M .

2) If we omit Δ we mean all first order formulas, if we omit m^* we mean ω . Also in $a_{u, \alpha, \ell}$ we may omit ℓ (it is $|u|$). Of course nothing changed if we allow $a_{u, \alpha, \ell}$ to be a finite sequence (with length depending on (α, ℓ) only).

3) We add “and over J ” where $J \subseteq I$ if in (*) we demand $(\forall x \in J) \bigwedge_{\ell} (x < t_\ell = x < s_\ell \ \& \ x = t_\ell \equiv x = s_\ell \ \& \ t_\ell < x \equiv x_\ell < x)$. We say “almost over J ” if we add $J \cap \{t_\ell : \ell < n\} = \emptyset$.

3.3 Claim. *[T has NIP] 1) Assume*

- (a) Δ is a finite set of formulas, $m^* < \omega$
- (b) M a model of T , $A \subseteq M$
- (c) $\mathbf{a} = \langle a_{u,k,\ell} : \ell < n, \alpha < k_\ell, u \in [I]^\ell \rangle$ is indiscernible over A
- (d) $\bar{d} \in {}^\omega M$.

Then there is a finite subset J of I such that $\langle a_{u,k,\ell} : \ell < n, k < k_\ell, u \in [I]^\ell \rangle$ is Δ -indiscernible over $A \cup \bar{d}$ almost over J .

2) Moreover, there is a bound on $|J|$ which depend just on $\Delta, \langle k_\ell : \ell < n \rangle$ (and T), and so it is enough that \mathbf{a} is Δ_1 -indiscernible for appropriate finite Δ_1 .

Proof. 1) Straightforward. If this fails, try to choose by induction on $i < \omega$, $\langle t_\ell^i : \ell < m_i \rangle, \langle s_\ell^i : \ell < m_i^* \rangle$ such that:

- (i) $m_i \leq m^*$
- (ii) $t_0^i < t_1^i < \dots < t_{m_i-1}^i, s_0^i < s_1^i < \dots < s_{m_i-1}^i$
- (iii) $t_m^i, s_m^i \notin J_i = \{t_\ell^j, s_\ell^j; j < i, \ell < m_j\}$
- (iv) $\langle t_\ell^i : \ell < m_i \rangle, \langle s_\ell^i : \ell < m_i \rangle$ exemplify that J_i is not as required.

Let $\bar{b}_i^0 = \langle a_{u,k,\ell} : \ell < n, k < k_\ell, u \in [\{t_0, \dots, t_{m_i-1}\}]^\ell \rangle$ and $b_i^1 = \langle a_{u,k,\ell} : \ell < n, k < k_\ell, u \in [\{s_0, \dots, s_{m_i-1}\}]^\ell \rangle$.

So clearly

- (*)₁ the Δ -types of $\bar{d} \wedge \bar{b}_i^0, \bar{d} \wedge b_i^{-1}$ over A are different
[why? by their choice]
- (*)₂ if $i(*) < \omega, \eta \in {}^{i(*)}2$, then the types of $\bar{b}_0^0 \wedge \bar{b}_1^0 \wedge \dots \wedge \bar{b}_{i(*)-1}^0$ and $\bar{b}_0^{\eta(0)} \wedge \bar{b}_1^{\eta(1)} \wedge \dots \wedge b_{i(*)-1}^{\eta(i(*)-1)}$ over A are equal
[why? by the indiscernibility].

So we are easily done. □_{3.3}

3.4 Claim. *[T has NIP] Assume $\bar{\mathbf{a}}^\ell = \langle \bar{a}_t^\ell : t \in I_\ell \rangle$ is an indiscernible sequence of cofinality $\kappa > |T|$ for $\ell = 1, 2$. Then we can find $s_i^\ell \in I_\ell$ for $\ell = 1, 2, i < \kappa$ such that $\langle a_{s_i^1}^1 \wedge \bar{a}_{s_i^2}^2 : i < \kappa \rangle$ is an indiscernible sequence.*

Proof. Easy by repeated use of 3.1.

3.5 Conclusion 1) Assume

- (*) $\langle \bar{b}_t : t \in I \rangle$ is an indiscernible sequence over A .

For every $\bar{c} \in {}^{\omega>}\mathfrak{C}$ there is $J \subseteq I$ of cardinality $\leq |T|$ and $\langle J_\varphi : \varphi \in L_{\tau(T)} \rangle, J_\varphi$ a finite subset of J such that:

- (*)₁ for every $\bar{a} \in {}^{\ell g(\bar{y})}A$ and $\varphi = \varphi(\bar{x}, \bar{y})$ there are $n \leq n_{\varphi(\bar{x}, \bar{y})}$ and $t_1 < \dots < t_n$ from J_φ such that if $r, s \in I \setminus \{t_1, \dots, t_n\}$ and $m \in [1, n] \Rightarrow s <_I t_m \equiv r <_I t_n$ then $\models \varphi[\bar{b}_s, \bar{a}] \equiv [\bar{b}_r, \bar{a}]$
- (*)₂ for every $k < \omega, \bar{a} \in {}^{\ell g(\bar{y})}A$ and $\varphi = \varphi(\bar{x}_1, \dots, \bar{x}_k, \bar{y})$ there are $n \leq n_\varphi$ and $t_1 < \dots < t_n$ from J_φ we have if $s_1 <_I \dots <_I s_k$ and $r_1 <_I \dots <_I r_k$ are from J and $m \in [1, n] \ \& \ \ell \in [1, k] \Rightarrow (s_\ell <_I t_m \equiv r_\ell <_I t_m) \ \& \ t_m <_I s_\ell \equiv t_m <_I r_\ell$ then $\models \varphi[\bar{b}_{s_1}, \dots, \bar{b}_{s_k}, \bar{a}] \equiv \varphi[\bar{b}_{r_1}, \dots, \bar{b}_{r_k}, \bar{a}]$.

2) Assume

- (*)₃ $\langle \bar{b}_{u, \alpha, \ell} : \ell < n, u \in [I]^n, \alpha < \alpha_\ell \rangle$ is indiscernible over A and $\alpha_\ell < \omega$ for $\ell < n$ (and $n < \omega$). For every \bar{c} there are $J \subseteq I, |J| \leq |T|$ and finite $J_\varphi \subseteq J$ for $\varphi \in L_{\tau(T)}$ such that the parallel of (*₁), (*₂) hold.

Proof. 1) Clearly (*₁) is a case of 3.3, if we apply it to $(\mathfrak{C}, a)_{a \in A}$. Similarly for (*₂) apply it enough times noting: if $\langle J_\ell : \ell \leq k+1 \rangle$ is an increasing sequence of subsets of I and $t, \dots, t_k \in I$ then for some $\ell < k+1, \{t_1, \dots, t_k\} \cap J_{\ell+1} \subseteq J_\ell$.

2) Similar.

Question: Can we find $\bar{b}_\alpha \in {}^{|T|}\mathfrak{C}$ such that $\langle \bar{b}_\alpha : \alpha < \lambda \rangle$ is an indiscernible sequence $\alpha \neq \beta \Rightarrow \bar{b}_\alpha \neq \bar{b}_\beta$ and for $\alpha < \beta < \gamma$ we have $\text{tp}(\bar{b}_\gamma, \bar{b}_\beta) \models \text{tp}(\bar{b}_\gamma, \bar{b}_\alpha)$?

Question: If $< (= \varphi(x, y, \bar{c}))$ is a partial order with infinite increasing sequences, we may consider κ -directed subsets, $\kappa = \text{cf}(\kappa > |T|)$, they define a Dedekind cut.

What about orthogonality of those?

§4 PERPENDICULAR ENDLESS INDISCERNIBLE SEQUENCES

Dimension and orthogonality play important role in [Sh:c], see in particular Ch.V. Now, as our prototype is the theory $\text{Th}(\mathbb{Q}, <)$, it is natural to look at cofinality, this is $\text{dual-cf}(\bar{\mathbf{b}}, A)$, measuring the cofinality of approaching $\bar{\mathbf{b}}$ from above (here $\bar{\mathbf{b}}$ is always indiscernible sequences with no last member). So a relative of orthogonality which we all perpendicularly suggest itself as relevant. It is defined in 4.3, as well as equivalence and dual-cf. Now perpendicularly is closely related to being mutual indiscernibility (see 4.4(1), 4.5(2), hence if T is unstable, then there are lost if pairwise perpendicular indiscernible sequences: $\text{cf}\langle \bar{a}_\alpha : \alpha < \lambda \rangle$ is an indiscernible sequence, not set and $\bar{\mathbf{b}}^\alpha = \langle \bar{a}_{\omega_\alpha+n} : n < \omega \rangle$ for $\alpha < \lambda$ then $\{\bar{\mathbf{b}}^\alpha : \alpha < \lambda\}$ are pairwise perpendicular. In this section we present basic properties of perpendicularly. In particular, it is preserved by equivalence (4.5(5)). For perpendicular sequences, we can more easily restrict them to get mutually indiscernible sets than in §3.

For indiscernible sets this essentially becomes orthogonality.

The case of looking at more than two indiscernible sequences reduced to looking at all pairs (4.7(2), 4.9(2)). Also, as in [Sh:c, V], if $\bar{\mathbf{b}}$ is not perpendicular to $\bar{\mathbf{a}}^\zeta$ for $\zeta < \zeta^*$ and the $\bar{\mathbf{a}}^\zeta$ -s are pairwise perpendicular then $\zeta^* < |T|^+$ (see ?).

Lastly, we recall the density of quite “types not splitting over small sets” (for theories with NIP), hence the existence of a “quite constructible” model over any A .

4.1 Definition. 1) We say the infinite sequences $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are mutually indiscernible if $\bar{\mathbf{b}}^\ell$ is indiscernible over $\cup\{\bar{b}_t^{3-\ell} : t \in \text{Dom}(\bar{\mathbf{b}}^{3-\ell})\}$ for $\ell = 1, 2$. Similarly over A . 2) We say that the family $\{\bar{\mathbf{b}}^\zeta : \zeta < \zeta^*\}$ of sequences is mutually indiscernible over A , if for $\zeta < \zeta^*$, $\bar{\mathbf{b}}^\zeta$ is indiscernible over $\cup\{\bar{b}_t^\varepsilon : \varepsilon \neq \zeta, \varepsilon < \zeta^*, t \in \text{Dom}(\bar{\mathbf{b}}^\varepsilon)\} \cup A$.

4.2 Hypothesis. T has NIP.

4.3 Definition. Let $\bar{\mathbf{a}}^\ell = \langle a_t^\ell : t \in I_\ell \rangle$ be an indiscernible sequence which are endless (i.e. I_ℓ having no last element) for $\ell = 1, 2$.

1) We say that $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are perpendicular when

(*) if \bar{b}_n^ℓ realizes $\text{Av}(\{\bar{b}_m^k : m < n \ \& \ k \in \{1, 2\} \vee m = n \ \& \ k < \ell\} \cup \bar{\mathbf{a}}^1 \cup \bar{\mathbf{a}}^2, \bar{\mathbf{a}}^\ell)$ for $\ell = 1, 2$ then $\bar{\mathbf{b}}^1, \bar{\mathbf{b}}^2$ are mutually indiscernible (see below) where $\bar{\mathbf{b}}^\ell = \langle \bar{b}_n^\ell : n < \omega \rangle$.

2) We say $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are equivalent if for every $A \subseteq \mathfrak{C}$ we have $\text{Av}(A, \bar{\mathbf{a}}^1) = \text{Av}(A, \bar{\mathbf{a}}^2)$. 3) If $\bar{\mathbf{a}}^1 \subseteq A$ we let $\text{dual-cf}(\bar{\mathbf{a}}^1, A) = \text{Min}\{|B| : B \subseteq A \text{ and no } \bar{c} \in {}^\omega A \text{ realizes } \text{Av}(B, \bar{\mathbf{a}}^1)\}$.

- 4.4 Claim.** 1) If \bar{a}^1, \bar{a}^2 are endless mutually indiscernible, then they are perpendicular.
- 2) “Mutually indiscernible” and “perpendicular” are symmetric relations.
- 3) On the family of endless indiscernible sequences, being equivalent is an equivalence relation.

Proof. Easy.

- 4.5 Claim.** 1) If $\bar{a}^\ell = \langle a_t^\ell : t \in I_\ell \rangle$ is an indiscernible sequence for $\ell = 1, 2$ and $|T| < \text{cf}(I_1), |I_1| < \text{cf}(I_2)$, then for some end segments J_1, J_2 of I_1, I_2 respectively, $\bar{a}^1 \upharpoonright J_1, \bar{a}^2 \upharpoonright J_2$ are mutually indiscernible.
- 2) If $\bar{a} = \langle a_t^\ell : t \in I_\ell \rangle$ is an indiscernible sequence for $\ell = 1, 2$ and $\text{cf}(I_1), \text{cf}(I_2)$ are infinite and distinct then \bar{a}^1, \bar{a}^2 are perpendicular.
- 3) If $\bar{a}^\ell = \langle a_t^\ell : t \in I_\ell \rangle$ is an endless indiscernible sequence for $\ell = 1, 2$, δ is limit ordinal and \bar{b}_α^ℓ realizes $\text{Av}(\{\bar{b}_\beta^k : \beta < \alpha \ \& \ k \in \{1, 2\} \text{ or } \beta = \alpha \ \& \ k < \ell\} \cup \bar{a}^1 \cup \bar{a}^2, \bar{a}^\ell)$ and $\bar{b}^\ell = \langle \bar{b}_\alpha^\ell : \alpha < \delta \rangle$ for $\ell = 1, 2$ then: \bar{a}^1, \bar{a}^2 are perpendicular iff \bar{b}^1, \bar{b}^2 are perpendicular.
- 4) If $\bar{a}^\ell = \langle a_t^\ell : t \in I_\ell \rangle$ is an endless indiscernible sequence and $J_\ell \subseteq I_\ell$ is unbounded for $\ell = 1, 2$, then \bar{a}^1, \bar{a}^2 are perpendicular iff $\bar{a}^1 \upharpoonright J_1, \bar{a}^2 \upharpoonright J_2$ are perpendicular.
- 5) If $\bar{a}^\ell = \langle a_t^\ell : t \in I^\ell \rangle$ are an endless indiscernible sequence for $\ell = 1, 2, 3, 4$ and \bar{a}^1, \bar{a}^3 are equivalent and \bar{a}^2, \bar{a}^4 are equivalent, then \bar{a}^1, \bar{a}^2 are perpendicular iff \bar{a}^3, \bar{a}^4 are perpendicular.

Proof. Straight.

Remark. 1) Replace \bar{a} by a sequence of concatenation of n -tuples from it (as in 2.2) preserve relevant properties.

2) In 4.5(1), can we weaken $|I_1| < \text{cf}|I_2|$ to $\text{cf}|I_1| \neq \text{cf}|I_2|$?

4.6 Claim. Assume $\bar{a}^\ell = \langle a_t^\ell : t \in I_\ell \rangle$ is an endless indiscernible sequence for $\ell = 1, 2$.

- 1) If \bar{a}^1 is an indiscernible sequence over A , then: \bar{a}^1 is an indiscernible set over A iff \bar{a}^1 is an indiscernible set over \emptyset .
- 2) \bar{a}^1 is nonstable in \mathfrak{C} iff \bar{a}^1 is nonstable in $(\mathfrak{C}, c)_{c \in A}$.
- 3) If \bar{a}^1, \bar{a}^2 are equivalent, then \bar{a}^1 is nonstable iff \bar{a}^2 is nonstable.
- 4) If $J_\ell \subseteq I_\ell$ is infinite and \bar{a}^1, \bar{a}^2 are mutually indiscernible then $\bar{a}^1 \upharpoonright J_1, \bar{a}^2 \upharpoonright J_2$ are mutually indiscernible over $\bigcup_{\ell=1}^2 (\bar{a}^\ell \upharpoonright (I_\ell \setminus J_\ell))$.

Proof. 1) The “only if” is trivial. For the other direction if $\bar{\mathbf{a}}^1$ is an indiscernible set over \emptyset but not over A , we easily get the independence property. I.e. assume that $t_0 < \dots < t_{n-1}$ in I_1 and π a permutation of $\{0, \dots, n-1\}$, $\varphi(\bar{b}, \bar{a}_{t_0}, \dots, \bar{a}_{t_{n-1}})$ & $\neg\varphi(\bar{b}, \bar{a}_{t_{\pi(0)}}, \dots, \bar{a}_{t_{\pi(n-1)}})$. Let $s_m \in I_1$ be pairwise distinct for $m < \omega$ so $\{\varphi(\bar{y}, \bar{a}_{s_n}, \dots, \bar{a}_{kn+n-1}) : k < \omega\}$ is an independent contradiction. [Saharon - details]

2) Follows.

3) Check directly. $\square_{4.6}$

4.7 Claim. 1) Assume $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ are as in 4.6. If $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2$ has cofinality $> |T|$ and are mutually indiscernible and $\bar{b} \in {}^{\omega>}\mathfrak{C}$, then for some end-segments J_1, J_2 of $\text{Dom}(\bar{\mathbf{a}}^1), \text{Dom}(\bar{\mathbf{a}}^2)$ respectively $\bar{\mathbf{a}}^1 \upharpoonright J_1, \bar{\mathbf{a}}^2 \upharpoonright J_2$ are mutually indiscernible over \bar{b} .
2) Assume $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2, \bar{\mathbf{a}}^3$ are endless indiscernible sequences and $I = \text{Dom}(\bar{\mathbf{a}}^\ell)$ and \bar{a}_t^ℓ realizes $\text{Av}(\{A_s^k : s <_I t \text{ \& } k \in \{1, 2, 3\} \cup s = t \text{ \& } k < \ell\}, \bar{\mathbf{a}}^\ell)$, then:

- (a) $\langle \bar{a}_t^1 \bar{a}_t^2 \bar{a}_t^3 : t \in I \rangle$ is an indiscernible sequence;
- (b) if $\bar{\mathbf{b}}_1$ is an indiscernible set then $\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2$ are mutually indiscernible
- (c) if any two of $\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2, \bar{\mathbf{a}}^3$ are mutually indiscernible and I_1, I_2, I_3 are disjoint unbounded subsets of I , then $\bar{\mathbf{a}}^1 \upharpoonright I_1, \bar{\mathbf{a}}^3 \upharpoonright I_3$ are mutually indiscernible over $\bar{\mathbf{a}}^2 \upharpoonright I_2$.

Proof. Similar to 4.6(1).

1) Assume not and let $I_\ell = \text{Dom}(\bar{\mathbf{a}}^\ell)$. We can choose by induction on $\zeta < |T|^+$ a tuple $(n_\zeta, u_0^\zeta, u_1^\zeta, u_2^\zeta, u_3^\zeta)$ such that:

- (*)₂(a) $|u_\ell^\zeta| = n_\zeta$
- (b) $u_0^\zeta \cup u_1^\zeta \subseteq I_1$
- (c) $u_2^\zeta \cup u_3^\zeta \subseteq I_2$
- (d) letting $\bar{a}^{\zeta, \ell}$ be $\langle \bar{a}_t^\zeta : t \in u_\ell^\zeta \rangle$ we have $\varphi_\zeta[\bar{a}^{\zeta, 0}, \bar{a}^{\zeta, 2}, \bar{b}]$ & $\neg\varphi_\zeta[\bar{a}^{\zeta, 1}, \bar{a}^{\zeta, 3}, \bar{b}]$
- (e) $\ell \in \{0, 1\}$ & $\varepsilon < \zeta$ & $s \in u_{2\ell}^\varepsilon \cup u_{2\ell+1}^\varepsilon$ & $t \in u_{2\ell}^\zeta \cup u_{2\ell+1}^\zeta \Rightarrow s <_{I_\ell} t$.

Now without loss of generality $n_\zeta = n^*, \varphi_\zeta = \varphi$ for $\zeta < |T|^+$. As $\varphi[\bar{a}^{\zeta, 0}, \bar{a}^{\zeta, 3}, \bar{b}]$ or $\neg\varphi[\bar{a}^{\zeta, 0}, \bar{a}^{\zeta, 3}, \bar{b}]$ without loss of generality $u_0^\zeta = u_1^\zeta$ or $u_2^\zeta = u_3^\zeta$, and by the symmetry without loss of generality the former. Now for every $\eta \in |T|^+ 2$ there is an elementary mapping $f_\eta, f_\eta \upharpoonright \bar{\mathbf{a}}^1$ the identity, f_η maps $\bar{a}^{\zeta, 2}$ to $\bar{a}^{\zeta, 2+\eta(\zeta)}$. Let g_η be an automorphism of \mathfrak{C} extending f_η^{-1} and let $\bar{b}_\eta = g_\eta(\bar{b})$. So $\varphi[\bar{a}^{\zeta, 0}, \bar{a}^{\zeta, 2}, \bar{b}_\eta]$ holds iff $\eta(\zeta) = 0$ so we are done having gotten a contradiction.

2) Without loss of generality I is dense with no complete interval and every interval has cardinality $> |T|$. Now

Clause (a):

Easy.

Clause (b):

For any $s_1 <_I \dots <_I s_{n-1}$, by the construction we know that: stipulating $s_0 = -\infty, s_n = +\infty$; $I_\ell = \{t \in I : s_\ell <_I t \leq_I s_{\ell+1}\}$, that the sequences $\bar{\mathbf{a}}^1 \upharpoonright I_0, \dots, \bar{\mathbf{a}}^1 \upharpoonright I_{n-1}$ are mutually indiscernible over $\bar{a}_{t_i}^2 \wedge \dots \bar{a}_{t_{m-1}}^2$. By 3.1 clause (c) we are done.

Clause (c):

- (*) if (I_1, I_2) are infinite disjoint intervals of I then $\bar{\mathbf{a}}^2 \upharpoonright I_2, \bar{\mathbf{a}}^3 \upharpoonright I_2$ are mutually indiscernible over $\bar{\mathbf{a}}^1 \upharpoonright I_1 \cup \{a_t^\ell : \ell = 1, 2, 3 \text{ and } t \in I \setminus I_1 \setminus I_2\}$
[why? by part (1), which we have already proved and (2)(a), i.e. the indiscernibility of $\langle \bar{a}_t^1 \bar{a}_t^2 \bar{a}_t^3 : t \in I \rangle$].

Well (*) holds under any permutation of $\{1, 2, 3\}$ so by the way the a_t^ℓ 's were chosen clearly we are done. $\square_{4.7}$

4.8 Claim. 1) If $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ are endless indiscernible, perpendicular, then for any $\varphi(\bar{x}, y, \bar{c})$ for some truth values \mathbf{t} we have:

- (a) for every large enough $s \in \text{Dom}(\bar{\mathbf{a}})$, for every large enough $t \in \text{Dom}(\bar{\mathbf{b}})$ we have $\mathfrak{C} \models \varphi(\bar{a}_s, \bar{b}_t, \bar{c})$
(b) for every large enough $t \in \text{Dom}(\bar{\mathbf{a}})$ for every large enough $s \in \text{Dom}(\bar{\mathbf{b}})$ we have $\mathfrak{C} \models \varphi[\bar{\mathbf{a}}_s, \bar{\mathbf{b}}_t, \bar{c}]$.

Proof. By 4.9(2).

4.9 Claim. 1) The parallel of 4.7 holds for several indiscernible sequences, that is, assuming $\bar{\mathbf{a}}^\zeta = \langle a_t^\zeta : t \in I_\zeta \rangle$ is an endless indiscernible sequence for $\zeta < \zeta^*$

- (A) If the intervals $[cf(I_\zeta), |I_\zeta|]$ are pairwise disjoint, $cf(I_\zeta) > |T| + \zeta^*$, then for some end segment J_ζ of I_ζ for $\zeta < \zeta^*$, we have $\langle \bar{\mathbf{a}}^\zeta \upharpoonright J_\zeta : \zeta < \zeta^* \rangle$ is mutually indiscernible, which means: each $\bar{\mathbf{a}}^\zeta \upharpoonright \mathbf{J}_\zeta$ is indiscernible over $\cup \{\bar{\mathbf{a}}^\varepsilon \upharpoonright J_\varepsilon : \varepsilon < \zeta^* \text{ \& } \varepsilon \neq \zeta\}$ (in fact we can get indiscernibility over $\cup \{\bar{\mathbf{a}}^\varepsilon : \varepsilon < \zeta^* \text{ \& } \varepsilon \neq \zeta\}$)
(B) Assume $\langle \bar{\mathbf{a}}^\zeta : \zeta < \zeta^* \rangle$ are mutually indiscernible, $\bar{b} \in {}^\omega \mathfrak{C}$ and $I_\zeta = \text{Dom}(\bar{\mathbf{a}}^\zeta)$ and $I_\zeta = cf(\text{Dom}(\bar{\mathbf{a}}^\zeta)) > |T| + \zeta^*$. Then there are end segments

J_ζ of I_ζ for $\zeta < \zeta^*$ such that $\langle \bar{\mathbf{a}}^\zeta \upharpoonright I_\zeta : \zeta < \zeta^* \rangle$ is mutually indiscernible over $\bar{\mathbf{b}}$.

- (C) If J is an infinite linear order disjoint to $\cup\{I_\zeta : \zeta < \zeta^*\}$ and a_t^ζ realizes $\text{Av}(\{\bar{a}_s^\varepsilon : \varepsilon < \zeta^* \text{ and } s \in I_\varepsilon \text{ or } s \in J \text{ \& } t <_J s \text{ or } s = t \text{ \& } \varepsilon < \zeta\}, \bar{\mathbf{a}}^\zeta)$ then $\{\langle \bar{a}_s^\zeta : s \in J \rangle : \zeta < \zeta^*\}$ are mutually indiscernible over $\cup\{\bar{a}_s^\varepsilon : \varepsilon < \zeta^*, s \in I_\varepsilon\}$.

2) We weaken in the conclusion the mutually indiscernible by mutually Δ -indiscernible, then we can weaken $\text{cf}(I_\zeta) > |T| + |\zeta^*|$ to $\text{cf}(I_\zeta) > |\zeta^*|$.

Proof. Easy.

4.10 Claim. 1) If $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence, $\bar{\mathbf{b}} \in {}^\omega > \mathfrak{C}$ then we can divide I to $\leq 2^{|T|}$ convex subsets $\langle I_\zeta : \zeta < \zeta^* \rangle$ such that $\langle \bar{\mathbf{a}} \upharpoonright I_\zeta : \zeta < \zeta^*, I_\zeta \text{ infinite} \rangle$ is mutually indiscernible over $\bar{\mathbf{b}}$.
2) Similarly in 4.9.

Proof. Easy.

4.11 Claim. Assume that

- (a) $\bar{\mathbf{b}}, \bar{\mathbf{a}}^\zeta$ are an endless indiscernible sequence for $\zeta < \zeta^*$
- (b) $\bar{\mathbf{a}}^\zeta, \bar{\mathbf{a}}^\varepsilon$ are perpendicular for $\zeta \neq \varepsilon$
- (c) $\bar{\mathbf{b}}, \bar{\mathbf{a}}^\zeta$ are not perpendicular.

Then $\zeta^* < |T|^+$.

Proof. Assume toward contradiction that $\zeta^* \geq |T|^+$. We let $A = \bar{\mathbf{b}} \cup \bigcup_{\zeta} \bar{\mathbf{a}}^\zeta$ and choose $\bar{a}_n^{\zeta^*,*}, \bar{b}_n^*$ for $n < \omega$ such that:

- (a) $\bar{a}_n^{\zeta^*,*}$ realized $\text{Av}(A \cup \{b_m^* : m < n\} \cup \{a_m^{\varepsilon,*} : m < n \text{ \& } \varepsilon < \zeta^* \text{ or } m = n \text{ \& } \varepsilon < \zeta\}, \bar{\mathbf{a}}^\zeta)$
- (b) \bar{b}_n^* realizes $\text{Av}(A \cup \{b_m^* : m < n\} \cup \{a_m^{\varepsilon,*} : m \leq n, \varepsilon < \zeta^*\}, \bar{\mathbf{b}})$.

For each ζ , as $\bar{\mathbf{b}}, \bar{\mathbf{a}}^\zeta$ are not perpendicular, we can find $n_\zeta < \omega, u_\zeta^\ell \in [\omega]^{n_\zeta}$ for $\ell = 0, 1, 2$ such that $\langle \bar{b}_n^* : n \in u_\zeta^0 \rangle \wedge \langle \bar{a}_n^{\zeta^*,*} : n \in u_\zeta^1 \rangle$ and $\langle \bar{b}_n^* : n \in u_\zeta^0 \rangle \wedge \langle \bar{a}_n^{\zeta^*,*} : n \in u_\zeta^2 \rangle$ does not realize the same type; say one satisfies $\varphi_\zeta(\bar{x}, \bar{y})$ the second not. As we can replace $\langle \bar{\mathbf{a}}^\zeta : \zeta < |T|^+ \rangle$ by any subsequence of length $|T|^+$, without loss of

generality $n_\zeta = n_*$, $u_\zeta^\ell = u_\ell$, $\varphi_\zeta = \varphi$. Now for every $\mathcal{U} \subseteq |T|^+$ let $f_\mathcal{U}$ be the elementary mapping with domain $\cup\{a_n^{\zeta,*} : n \in u_1, \zeta < |T|^+\}$, mapping $a_{n_1}^{\zeta,*}$ to $a_{n_2}^{\zeta,*}$ iff $\zeta \in \mathcal{U}$, $n_1 = n_2$ or $\zeta \in |T|^+ \setminus \mathcal{U}$, $n_1 \in u_1$, $n_2 \in u_2$, $|n_2 \cap u_1| = |n_2 \cap u_2|$. Let $g_\mathcal{U}$ be an automorphism of \mathfrak{C} extending $f_\mathcal{U}^{-1}$. We have gotten the independence property for $\varphi(\bar{x}, \bar{y})$ as $g_\mathcal{U}(\langle \bar{b}_n^* : n \in u_0^0 \rangle)$ realizes $\{\varphi(\langle \bar{x}_n : n \in u_0 \rangle, \langle \bar{b}_n^{\zeta,*} : n \in u_1 \rangle)^{\text{if}(\zeta \in \mathcal{U})} : \zeta < |T|^+\}$, contradiction. $\square_{4.11}$

* * *

Recall ([Sh:c, Ch.III, §7])

- 4.12 Definition.** 1) $p \in \mathbf{F}_\kappa^{sp}(B)$ if for some set A we have $p \in S^{<\omega}(A)$, $B \subseteq A$, $|B| < \kappa$ and p does not split over B .
 2) $\mathcal{A} = (A, \langle \bar{b}_i, B_i : i < i^* \rangle)$ is an \mathbf{F}_κ^{sp} -construction (or $\langle b_i, B_i : i < i^* \rangle$ is an \mathbf{F}_κ^{sp} -construction over A) if $\text{tp}(\bar{b}_i, A \cup \{b_j : j < i\}) \in F_\kappa^{sp}(B_i)$, so $B_i \subseteq A_i^\mathcal{A} =: A \cup \{b_j : j < i\}$.
 3) Omitting B_i means for some B_i ; let $i^* = \ell g(\mathcal{A})$.

Remark. We may use \bar{b}_i 's of any length $< \kappa$.

- 4.13 Claim.** 1) If $B \subseteq A$, p is an m -type over B , then there is $q \in S^m(A)$ extending p and $B_1 \subseteq A$, $|B_1| \leq |T|$ such that q does not split over $B \cup B_1$.
 2) For any A and $\kappa = \text{cf}(\kappa) > |T|$ there is a model M and \mathbf{F}_κ^{sp} -construction $\mathcal{A} = (A, \langle \bar{b}_i, B_i : i < i^* \rangle)$ such that:

- (a) $M = A_{i^*}^\mathcal{A}$, $\|M\| = |A|^{<\kappa} + \sum_{\theta < \kappa} 2^{2^\theta}$
- (b) M is κ -saturated, moreover if $B \subseteq M$, $|B| < \kappa$, $p \in S^m(M)$ does not split over B then for unboundedly many $i < i^*$, \bar{b}_i realizes $p \upharpoonright A_i^\mathcal{A}$
- (c) $\text{cf}(i^*) \geq \kappa$

- 3) If \mathcal{A} is an \mathbf{F}_κ^{sp} -construction, $\kappa = \text{cf}(\kappa)$, $\bar{b} \subseteq A_{\ell g(\mathcal{A})}^\mathcal{A}$ has length $< \kappa$, then $\text{tp}(\bar{b}, A)$ does not split over some $B \subseteq A$, $|B| < \kappa$.

§5 INDISCERNIBLE SEQUENCE PERPENDICULAR TO CUTS

Our aim is to show that for a set of $\{\bar{\mathbf{b}}_\zeta : \zeta < \zeta^*\}$ of pairwise perpendicular endless indiscernible sets, we can find a model $M \supseteq \cup\{\bar{\mathbf{b}}_\zeta : \zeta < \zeta^*\}$ with $\langle \text{dual-cf}(\bar{\mathbf{b}}_\zeta) : \zeta < \zeta^* \rangle$ essentially as we like, and other $\bar{\mathbf{b}}'$ in M has such dual cofinality iff this essentially follows. Toward this we define and investigate when an endless indiscernible sequence $\bar{\mathbf{c}}$ is perpendicular to a (Dedekind) cut (I_1, I_2) is an indiscernible sequence $\bar{\mathbf{a}}$.

5.1 Definition. 1) We say (I_1, I_2) is a Dedekind cut of the linear order I , if I is the disjoint union of I_1, I_2 and $s \in I_1 \ \& \ t \in I_2 \Rightarrow s <_I t$ and we write $I, I = I_1 + I_2$, and its cofinality is $(\text{cf}(I_1), \text{cf}(I_2^*))$. If I is a convex subset of J and $I_1 \neq \emptyset \neq I_2$ we may abuse our notation saying “ (I_1, I_2) is a Dedekind cut of J ”. We say (I_1, I_2) is a Dedekind cut of $\bar{\mathbf{a}}$ if it is a Dedekind cut of $\text{Dom}(\bar{\mathbf{a}})$. If not say otherwise, $I_1 \neq \emptyset \neq I_2$, and the cut is nontrivial if both its cofinalities are infinite.

2) $(J_1, J_2) \leq (I_1, I_2)$ if J_1 is an end segment of I_1 and J_2 is an initial segment of I_2 .

3) We say the set A respects the Dedekind cut (I_1, I_2) of $\bar{\mathbf{a}}$ if (I_1, I_2) is a Dedekind cut of $\bar{\mathbf{a}}$ and for every $\bar{b} \in {}^\omega A$ for some $(J_1, J_2) \leq (I_1, I_2)$ the sequence $\bar{\mathbf{a}} \upharpoonright (J_1 + J_2)$ is indiscernible over \bar{b} .

4) For endless indiscernible sequences $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ and Dedekind cut (I_1, I_2) of $\bar{\mathbf{a}}$ we say that $\bar{\mathbf{b}}$ is perpendicular to the cut when: if $\bar{\mathbf{b}}'$ is an indiscernible sequence over $\bar{\mathbf{b}} \cup \bar{\mathbf{a}}$ based on $\bar{\mathbf{b}}$ (see below) then $\bar{\mathbf{b}}' \cup \bar{\mathbf{a}}$ respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$.

5) For endless indiscernible sequences $\bar{\mathbf{a}}$ and $A \supseteq \bar{\mathbf{a}}$ we say an endless indiscernible sequence $\bar{\mathbf{b}} = \langle \bar{b}_i : i \in I \rangle$ over A is based on $\bar{\mathbf{a}}$ if each \bar{b}_t realizes $\text{Av}(A \cup \{\bar{b}_s : s <_I t\}, \bar{\mathbf{a}})$.

5.2 Claim. 1) If $\langle A_i : i < \delta \rangle$ is increasing, $\bar{\mathbf{a}} \subseteq A_0$ is an endless indiscernible sequence, $\bar{a}'_i \subseteq A_{i+1}$ realizes $\text{Av}(A_i, \bar{\mathbf{a}})$, $\bar{\mathbf{a}}' = \langle \bar{a}'_i : i < \delta \rangle$, $\bar{\mathbf{a}}''$ is the inverse of $\bar{\mathbf{a}}'$ then

(a) $\bar{\mathbf{a}} \hat{\ } \bar{\mathbf{a}}''$ is indiscernible

(b) the set $\bigcup_{i < \delta} A_i$ respects the cut $(\text{Dom}(\bar{\mathbf{a}}), \text{Dom}(\bar{\mathbf{a}}''))$ of $\bar{\mathbf{a}} \hat{\ } \bar{\mathbf{a}}''$.

2) If $\bar{\mathbf{a}}$ is a non stable indiscernible sequence, $\bar{\mathbf{a}} \subseteq A$, the set A respects the endless cut (I_1, I_2) of $\bar{\mathbf{a}}$ and the cofinalities of the cut are $> |T|$ then $\text{dual-cf}(\bar{\mathbf{a}} \upharpoonright I_1, M) = \text{cf}(I_2^*)$.

3) If $\bar{\mathbf{a}}$ is an indiscernible sequence with Dedekind cut (I_1, I_2) of cofinality (κ_1, κ_2) , $\aleph_0 \leq \kappa_1, \kappa_2$ and $\bar{\mathbf{c}}$ an endless indiscernible sequence respecting this cut then: for some for every formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ and sequence \bar{b} for some truth value \mathbf{t} we have:

(*) (i) for every large enough $s \in \text{Dom}(\bar{\mathbf{c}})$, for some $(J_1, J_2) \leq (I_1, I_2)$ for every $t \in J_1 \cup J_2$ we have $\mathfrak{C} \models \varphi[\bar{a}_2, \bar{b}, \bar{c}_1]^\mathbf{t}$,

- (ii) for some $(J_1, J_2) \leq (I_1, I_2)$ for every $t \in J_1 \cup J_2$ we have $\mathfrak{C} \models \varphi[\bar{a}_2, \bar{b}, \bar{c}_1]^t$,
 ??? and end segment J .

4) If in part (3), $|A| + |A| < \kappa_1, \kappa_2$ then for some $(J_1, J_2) \leq (I_1, I_2)$ we have: of $\text{Dom}(\bar{\mathfrak{c}}), \bar{\mathfrak{a}} \restriction (J_1 + J_2), \bar{\mathfrak{c}} \restriction J$ are mutually Δ -indiscernible.

Proof. 1), 2) Straightforward.

3) Let $\delta = |T|^+$, \bar{c}_γ realizes $\text{Av}(\bar{\mathfrak{a}} \cup \bar{\mathfrak{c}} \cup \{\bar{c}_\beta : \beta < \gamma\}, \bar{\mathfrak{c}})$, for $\gamma < \delta$ so by the definition of “respect the Dedekind cut”, there is $(J_1, J_2) \leq (I_1, I_2)$ such that $\bar{\mathfrak{a}} \restriction (J_1 \cup J_2), \langle \bar{c}_\gamma : \gamma < \delta \rangle$ are mutually indiscernible. Let (I_1, I_2) have cofinality κ_1, κ_2 and for our purpose without loss of generality $\kappa_1, \kappa_2 > |T|$. Now $\bar{\mathfrak{a}} \restriction J_1$, the inverse of $\bar{\mathfrak{a}} \restriction J_2, \langle \bar{c}_\gamma : \gamma < \delta \rangle$ are mutually indiscernible, hence by 4.9, clause (B) without loss of generality they are mutually indiscernible over \bar{b} (i.e. omitting an initial segment of each and renaming. So we have truth values $\mathfrak{t}(1), \mathfrak{t}(2)$ such that $t \in J_\ell$ & $\gamma < \delta \Rightarrow \mathfrak{C} \models \varphi[\bar{a}_t, \bar{b}, \bar{c}_\gamma]^{\mathfrak{t}(\ell)}$. If $\mathfrak{t}(1) \neq \mathfrak{t}(2)$ we get contradiction to “ T has NIP” so without loss of generality $\mathfrak{t}(1) = \mathfrak{t}(2)$ and as we can replace φ by $\neg\varphi$ without loss of generality $\mathfrak{t}(1) = \mathfrak{t}(2) = \text{truth}$. So by the choice of $\langle \bar{c}_\gamma : \gamma < \delta \rangle$, for every $t \in J_1 \cup J_2$, for every large enough $s \in \text{Dom}(\bar{\mathfrak{c}})$ we have $\mathfrak{C} \models \varphi[\bar{a}_t, \bar{b}, \bar{c}_s]$.

Clearly $\bar{\mathfrak{c}}, \bar{\mathfrak{a}} \restriction J_1$ is perpendicular (by 4.8).

5.3 Claim. Assume

- (a) $I = I_1 + I_2$ and the Dedekind cut (I_1, I_2) has cofinality (κ_1, κ_2)
- (b) $\bar{\mathfrak{a}} = \langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence
- (c) $\bar{\mathfrak{a}} \subseteq A$
- (d) the set A respects the cut (I_1, I_2) of $\bar{\mathfrak{a}}$.

1) If $\text{tp}(\bar{b}, A) \in \mathbf{F}_\kappa^{sp}$ and $\kappa \leq \kappa_1, \kappa_2$, then the set $A \cup \bar{b}$ respects the cut (I_1, I_2) of $\bar{\mathfrak{a}}$. Assume in addition

- (e) $|T| < \kappa_1, \kappa_2$
- (A) If $\bar{\mathfrak{c}} \subseteq A$ is an endless indiscernible sequence and $\bar{\mathfrak{c}}$ is perpendicular to the cut (I_1, I_2) of $\bar{\mathfrak{a}}$ and \bar{c} realizes $\text{Av}(A, \bar{\mathfrak{c}})$, then $A \cup \bar{c}$ respects the cut (I_1, I_2) of $\bar{\mathfrak{a}}$.
- (B) If $A_i (i < \delta)$ is increasing each A_i respects the cut (I_1, I_2) of $\bar{\mathfrak{a}}$ then also $\bigcup_{i < \delta} A_i$ does.

2) If $A^+ = A \cup \{a_i : i < i^*\}$ and for each i , $tp(a_i, A \cup \{a_j : j < i^*\})$ belongs to $\mathbf{F}_{\min\{\kappa_1, \kappa_2\}}^{sp}$ or is $\text{Av}(A \cup \{a_j : j < i\}, \bar{\mathbf{b}})$ where $\bar{\mathbf{b}} \subseteq A \cup \{a_j : j < i\}$ is an endless indiscernible sequence perpendicular to the cut (I_1, I_2) of $\bar{\mathbf{a}}$, then A^+ respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$.

Proof. 1) Check.

2) Suppose that this fails, so there is $b \in {}^{\omega>}(A)$ such that $\bar{b} \hat{\ } \bar{c}$ witness it. Now by assume (e) for some $(J_1, J_2) \leq (I_1, I_2)$ we have $\bar{a} \upharpoonright J_1, \bar{\mathbf{a}} \upharpoonright J_2$ are mutually indiscernible over $\bar{b} \hat{\ } \bar{c}$. As $\bar{b} \hat{\ } \bar{c}$ witness failure for some $\varphi(\bar{x}, \bar{y}, \bar{z})$ we have

- $(*)_1(\alpha)$ $t \in J_1 \Rightarrow \mathfrak{C} \models \varphi[\bar{a}_t, \bar{b}, \bar{c}]$ and
 (β) $t \in J_2 \Rightarrow \mathfrak{C} \models \neg \varphi[\bar{a}_t, \bar{b}, \bar{c}]$.

By clause (α) , for every $t \in J_1$ for every large enough $s \in \text{Dom}(\bar{c})$ we have $\mathfrak{C} \models \varphi[\bar{a}_t, \bar{b}, \bar{c}_s]$, however, \bar{c} is perpendicular to the cut (I_1, I_2) of $\bar{\mathbf{a}}$ hence for every large enough $s \in \text{Dom}(\bar{c})$ for every large enough $t_1 \in J_1$ and small enough $t_2 \in J_2$ we have $\mathfrak{C} \models \varphi[\bar{a}_{t_1}, \bar{b}, \bar{c}_s]$ & $\varphi[\bar{a}_{t_2}, \bar{b}, \bar{c}_s]$. Again as \bar{c} is perpendicular to the cut (I_1, I_2) of $\bar{\mathbf{a}}$ we get: for every small enough $t_2 \in J_2$ for every large enough $s \in \text{Dom}(\bar{c})$ we have $\mathfrak{C} \models \varphi[\bar{a}_{t_2}, \bar{b}, \bar{c}_1]$, contradicting clause (β) of $(*)$.

3) Check the definition.

4) Prove by induction on i using (1), (2), (3). $\square_{5.3}$

5.4 Claim. Assume

- (a) (I_1, I_2) is a cut of the indiscernible sequence $\bar{\mathbf{a}}$ with both cofinalities infinite
- (b) $\bar{\mathbf{b}}$ is an endless indiscernible sequence
- (c) $\bar{\mathbf{a}} \upharpoonright I_1, \bar{\mathbf{b}}$ are perpendicular
- (d) for $t \in I_2, \bar{a}_t^1$ realizes $\text{Av}(\{\bar{a}_s^1 : s \in I_1 \vee t <_I s \in I_2\} \cup \bar{\mathbf{b}}, \bar{\mathbf{a}}^1 \upharpoonright I_1)$.

Then $\bar{\mathbf{b}}$ is perpendicular to the cut (I_1, I_2) of $\bar{\mathbf{a}}$.

Proof. First assume that the cofinalities of $I_1, I_2^*, \bar{\mathbf{b}}$ are $> |T|$. By the demands (c) + (d), $\bar{\mathbf{a}} \cup \bar{\mathbf{b}}$ respect the cut (I_1, I_2) of $\bar{\mathbf{a}}$. So assume $A \supseteq \bar{\mathbf{a}} \cup \bar{\mathbf{b}}$ respect the cut (I_1, I_2) of $\bar{\mathbf{a}}$ and \bar{b} realizes $\text{Av}(A, \bar{\mathbf{b}})$. So let $\bar{c} \in {}^{\omega>}A$; as A respects the cut (I_1, I_2) of $\bar{\mathbf{a}}$, there is $(J_1, J_2) \leq (I_1, I_2)$ such that $\bar{\mathbf{a}} \upharpoonright (J_1 + J_2)$ is indiscernible over \bar{c} and no problem.

Generally deal with Δ -types for finite Δ 's. $\square_{5.4}$

5.5 Claim. *Assume*

- (a) $\lambda = \lambda^{<\kappa_2}$
- (b) $\kappa_1 = \text{cf}(\kappa_1) \leq \kappa_2 \leq \theta_2 = \text{cf}(\theta_2), \kappa_1 \leq \theta_1 = \text{cf}(\theta_1) \leq \lambda$
- (c) $|A| \leq \lambda$
- (d) $\bar{\mathbf{a}}^\zeta \subseteq A$ is endless, nonstable indiscernible for $\zeta < \zeta^*$ and $\zeta^* \leq \lambda$
- (e) the $\bar{\mathbf{a}}^\zeta$ for $\zeta < \zeta^*$ are pairwise perpendicular.

Then we can find a model M such that

- (α) $A \subseteq M$
- (β) $\text{dual-cf}(\bar{\mathbf{a}}^\zeta, M) = \theta_1$ for every $\zeta < \zeta^*$
- (γ) if $\bar{\mathbf{a}} \subseteq M$ is a nonstable endless indiscernible sequence of cardinality (hence cofinality) $< \kappa_2$ perpendicular to every $\bar{\mathbf{a}}^\zeta$ then $\text{dual-cf}(\bar{\mathbf{a}}, M) = \theta_2$
- (δ) M is κ_1 -saturated.

Proof. We first deal with a restricted case, then derive from it the general case.

Case 1: $|A| < \lambda = \text{cf}(\lambda) = \theta_1, \kappa_1 = \kappa_2 = \text{cf}(\kappa_2) \leq \theta_2$ and $(\forall \alpha < \lambda)(|\alpha|^{\theta_2} < \lambda)$ and in clause (γ) we demand just $\text{dual-cf}(\bar{\mathbf{a}}, M) \in [\theta_2, \lambda)$.

We can find a_i for $i < \lambda$ such that letting $A_i = A \cup \{a_j : j < i\}$ we have

- (i) for each $i < \lambda$ we have $\text{tp}(\bar{a}_i, A_i) \in \mathbf{F}_{\theta_2}^{sp}$ or $\text{tp}(a_i, A_i) = \text{Av}(A_i, \bar{\mathbf{a}}^\zeta)$ for some $\zeta < \zeta^*$
- (ii) if $p \in S^{<\omega}(A_\lambda), p \in \mathbf{F}_{\theta_2}^{sp}$ then for λ ordinals $j < \lambda, p$ is realized by b_j .

This is straightforward and clauses (α), (β), (δ) obviously hold. As for clause (γ)⁻, let $\bar{\mathbf{a}} = \langle a_t : t \in I \rangle \subseteq M$ endless indiscernible $|I| < \theta_2, \bar{\mathbf{a}}$ perpendicular to every $\bar{\mathbf{a}}^\zeta$ and assume toward contradiction that $\text{dual-cf}(\bar{\mathbf{a}}, M) \notin [\theta_2, \lambda)$ but trivially it is $\leq \|M\| = \lambda$ and by saturation $\geq \theta_2$, so necessarily it is λ .

So for every $\alpha < \lambda$ some $\bar{c}_\alpha \in M$ realizes $\text{Av}(A_\alpha, \bar{\mathbf{a}})$ and let $\beta_\alpha = \text{Min}\{\beta < \lambda : \bar{c}_\alpha \subseteq A_\beta\}$. So $\beta_\alpha \in (\alpha, \lambda)$ and let $E = \{\delta < \lambda : \delta \text{ limit } \bar{\mathbf{a}} \subseteq A_\delta \text{ and } (\forall \alpha < \delta), \beta_\alpha < \delta\}$. For $\delta \in \text{acc}(E)$ let $\bar{a}^{\delta,0}$ be the sequence $\langle \bar{c}_\alpha : \alpha \in E \cap \delta \rangle$ and $\bar{\mathbf{a}}^\delta$ be its inverse. Now not only is $\bar{\mathbf{a}} \hat{\ } \bar{\mathbf{a}}^\delta$ is indiscernible but A_δ respects the cut $(\text{Dom}(\bar{\mathbf{a}}), \text{Dom}(\bar{\mathbf{a}}^\delta))$ of $\bar{\mathbf{a}}$ to 5.2(2). Choose $\delta(*) \in \text{acc}(E)$ be of cofinality θ_2 . Now by 5.3(4) $\bar{\mathbf{a}} \hat{\ } \bar{\mathbf{a}}^{\delta(*)}$ is respected also by $A_\lambda = M$. By 5.2(2) this gives $\text{dual-cf}(\bar{\mathbf{a}}, M) = \text{cf}(\delta(*)) = \theta_2$, contradiction.

Case 2: As above getting the full (γ) . By absoluteness arguments without loss of generality we can find θ_*, λ_* such that $\theta_* = \theta_*^{<\theta_*}, 2^{\theta_*} = \lambda_*$ and they are $> \lambda$. Now use case A for θ_*, λ_* getting M_0 . We can find M such that

- (*)(a) $M_1 \prec M_0$ include A and has cardinality θ_*, κ -saturated
- (b) for $\zeta < \zeta^*$, $\text{dual-cf}(\bar{\mathbf{a}}^\zeta, M_1) = \theta_1$
- (c) any endless nonstable indiscernible sequence $\bar{\mathbf{a}} \subseteq M_1, |\text{Dom}(\bar{\mathbf{a}})| < \theta_2$ perpendicular to every $\bar{\mathbf{a}}^\zeta$, $\text{dim-cf}(\bar{\mathbf{a}}) = \theta_2$
 [why it exists? we choose by induction on $\alpha < \theta_1, M_{1,\alpha}$ satisfying clause (*)(a), increasing continuous with α such that if $\bar{\mathbf{a}} \subseteq M_{1,\alpha}$ is as in (2), then a witness to $\text{dim-cf}(\bar{\mathbf{a}}, M) = \theta_2$ is included in $M_{1,\alpha+1}$ and $\text{Av}(M_{1,\alpha}, \bar{\mathbf{a}}^\zeta)$ is realized in $M_{1,\alpha+1}$. Now $\bigcup_{\alpha < \theta_1} M_{1,\alpha}$ is as required.

Now similarly we can find $M_2 \prec M_1$ as required this time by a sequence θ_2 approximations. $\square_{5.5}$

§6 CONCLUDING REMARK

6.1 Discussion: A major lack of this work is the absence of test questions.

A candidate is (see [Sh 702, §2]).

6.2 Question: If $A \subseteq \mathfrak{C}_T$, $\kappa = |A| + |T|$ (or $\kappa = \beth_7(|A| + |T|)$) and $\lambda = \beth(2^\kappa)^+$ (or larger, but no large cardinals) and $a_i \in \mathfrak{C}_T$ for $i < \lambda$ then for some $w \in [\lambda]^{\kappa^+}$, the sequence $\langle a_i : i \in \omega \rangle$ is an indiscernible sequence over A (in \mathfrak{C}_T).

Through this property does not characterize NIP, it is quite natural in this context. See also next.

Another direction is generalizing DOP, which in spite of its name is a non first order independence property.

6.3 Definition. T has the dual-cf- κ -dimensional independence if: $\bar{\kappa} = (\kappa_0, \kappa_1, \kappa_2)$, $\kappa_1 \neq \kappa_2 > \kappa_0 < \kappa_1, \kappa_0 < \kappa_2$ and for every λ and $R \subseteq \lambda \times \lambda$ symmetric we can find $M_R, \bar{\mathbf{b}}_\alpha, \bar{\mathbf{c}}_\alpha \in {}^{\kappa_0}(M_R)$ and $\mathbf{I}_{\alpha,\beta} = \langle \bar{a}_{\alpha,\beta,i} : i < \kappa_0 \rangle \subseteq M_R$ for $(\alpha, \beta) \in R, \alpha < \beta$ such that:

- (a) the type of $\bar{\mathbf{b}}_\alpha \wedge \bar{\mathbf{c}}_\beta \wedge \mathbf{I}_{\alpha,\beta}$ is the same for all α, β
- (b) $\text{dual-cf}(\mathbf{I}_{\alpha,\beta}, M_R) = \kappa_1$
- (c) if $\alpha < \beta \neg \alpha R \beta$, $\mathbf{I}'_{\alpha,\beta} = \langle a'_{\alpha,\beta,i} : i < \kappa_0 \rangle \subseteq M_R$ is such that for every $(\alpha_1, \beta_1) \in R$ there is an automorphism h of \mathfrak{C} taking $\bar{\mathbf{b}}_{\alpha_1}$ to $\bar{\mathbf{b}}_\alpha$, $\bar{\mathbf{c}}_{\beta_1}$ to $\bar{\mathbf{c}}_\beta$ and $\bar{a}_{\alpha_1,\beta_1}$ to $a_{\alpha,\beta,i}$ then $\text{dual-cf}(\mathbf{I}'_{\alpha,\beta}, M) = \kappa_2$
- (d) M_R is κ_0^+ -saturated.

Note that (d), (d) follows from

- (c)⁺ if $\mathbf{I}'_{\alpha,\beta} = \langle \bar{a}'_{\alpha,\beta,i} : i < \kappa_0 \rangle$ realizes the relevant type $(\alpha, \beta) \notin R, \alpha < \beta < \lambda$ and $\alpha_1 < \beta_1 < \lambda, (\alpha_1, \beta_1) \in R$ then $\mathbf{I}'_{\alpha,\beta}, \mathbf{I}_{\alpha_1,\beta_1}$ are perpendicular.

As in §5 we can show that many variants are equivalent (using $+\infty, -\infty$ to absorb). We can similarly discuss deepness.

6.4 Discussion: 1) It is know that e.g. the p -adics are NIP (but unstable). Does this work tell us anything on them? Well, the construction in §5 gives somewhat more than what unstability gives: complicated models with more specific freedom. Note that instead $\text{dual-cf}(\mathbf{I}, M)$ we can use more complicated invariants (see [Sh:e, Ch.III,§3]) or earlier works).

We can, of course, (for the p -adic) characterize directly when indiscernible sequences are perpendicular.

2) We may like to define super-NIP (and $\kappa_{\text{nip}}(T)$) (parallel of superstable, i.e. $\kappa(T) = \aleph_0$ or super simple $\kappa_{\text{cst}}(T) = \aleph_0$). This is not clear to me. We may try the definition “ $w(\mathbf{I}) < \aleph_0$ ” for every endless indiscernible sequence where

6.5 Definition. For an endless indiscernible sequence \mathbf{I} let $w(\mathbf{I}) = \sup\{\alpha : \text{there is a sequence of length } \alpha \text{ of pairwise endless indiscernible sequences each non perpendicular to } \mathbf{I}\}$. But $w(\mathbf{I})$ is not exactly like dimension in the sense of algebraic manifolds.

Question: Assume $\mathbf{I}_\ell = \langle a_t^\ell : t \in I_\ell \rangle$ for $\ell = 1, 2$ are endless indiscernible nonperpendicular sequences

- (a) find a definable equivalence relation E such that $\langle a_t^2/E : t \in I_2 \rangle$ is nontrivial and $a_t^2 \in \text{acl}(\mathbf{I}_1 \cup \{a_s^2 : s <_{I_2} t\})$ for any large enough t
- (b) if $(\mathbf{I}_1, \mathbf{I}_2)$ is $(1, < \omega)$ -mutual indiscernible, can we define a derived group? More generally, it seems persuasive that groups appear naturally, particularly ordered groups
- (c) does the fact that putting of elements together, make strong splitting to dividing helps?
- (d) can the canonical bases of §1 help? Do they help for simple theories
- (e) what can we say on “ $\bar{\mathbf{a}}$ orthogonal to a set model A ?”

Can we say more on “stable” aspects? (see §1).

Cherlin wonders on the place of parallel algebraic geometric dimension and place of 0-minimal theory. In my perception probably if we succeed in 2), we may have a minimality notion which may then be characterized as some cases, but maybe it does not fit.

6.6 Question: Given two non perpendicular types which are weakly perpendicular can we find naturally defined groups?

* * *

6.7 Claim. *Assume*

- (α) $\bar{\mathbf{b}}^0 = \langle \bar{b}_t : t \in I_0 \rangle$ is an infinite indiscernible sequence over A
- (β) $B \subseteq \mathfrak{C}$.

Then we can find I_1 and \bar{b}_t for $t \in I_1 \setminus I_0$ such that:

- (a) $I_0 \subseteq I_1, |I_1 \setminus I_0| \leq |B| + |T|$

- (b) $\bar{\mathbf{b}}' = \langle \bar{b}_t : t \in I_1 \rangle$ is an indiscernible sequence over A
- (c) if $I_1 \subseteq I_2$ and \bar{b}_t for $t \in I_2 \setminus I_1$ are such that $\bar{\mathbf{b}}^2 = \langle \bar{b}_t : t \in I_1 \rangle$ is an indiscernible sequence over A .

Proof. We try to choose by induction on $\zeta < \lambda^+$ where $\lambda = |T| + |B|$ a sequence $\bar{b}^\zeta = \langle b_t : t \in J_\zeta \rangle$ and $\eta_\zeta, \bar{s}_\zeta, \bar{t}_\zeta, J^*, \varphi_\zeta, \bar{c}_\varepsilon, \bar{d}_\varepsilon$

- (a) J_ζ is a linear order, increasing continuous with ζ
- (b) $J_0 = I_0, J_{\varepsilon+1} \setminus J_\varepsilon$ is finite
- (c) \bar{b}^ζ is an indiscernible sequence over A
- (d) if $\zeta = \varepsilon + 1$ then $n_\varepsilon < \omega, \bar{s}_\varepsilon \in {}^{n_\varepsilon}(J_\zeta), \bar{t}_\varepsilon \in {}^{n_\varepsilon}(J_\zeta), \varphi_\varepsilon = \varphi_\varepsilon(\bar{x}_0, \dots, \bar{x}_{n_\varepsilon}, \bar{c}_\varepsilon, \bar{d}_\varepsilon), \bar{c}_\varepsilon \subseteq B, \bar{d}_\varepsilon \subseteq A$ and $J_\varepsilon^* = \cup \{ \bar{s}_\xi \hat{\ } \bar{t}_\xi : \xi < \varepsilon \}$
- (e) $\bar{s}_\varepsilon \sim_{J_\varepsilon^*} \bar{t}_\varepsilon$ and $\models \varphi[\bar{b}_{\bar{s}_\varepsilon}, \bar{c}_\varepsilon, \bar{d}_\varepsilon] \ \& \ \neg \varphi[\bar{b}_{\bar{t}_\varepsilon}, \bar{c}_\varepsilon, \bar{d}_\varepsilon]$ where $\bar{b}_{\langle t_\ell : \ell < n \rangle} = \bar{b}_{t_0} \hat{\ } \bar{b}_{t_1} \hat{\ } \dots \hat{\ } \bar{b}_{t_{n-1}}$.

If we succeed, without loss of generality $n_\varepsilon = n_*, \varphi_\varepsilon = \varphi_*, \text{bar } c_\varepsilon = \bar{c}^*$, and we get contradiction to 3.1. If we are stuck at stage ε , then $\bar{\mathbf{b}}^\varepsilon$ is as required.

Concluding Remark. We can define when an endless indiscernible sequence is orthogonal to a set and the dimensional independence property and prove natural properties, we intend to pursue this.

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